Trig - Identities

Relationships

SPTA Mathematics – Topic Questions with Notes

Formula. These formulae <u>are **not** given</u> on the National 5 Mathematics exam paper.

$$\tan x = \frac{\sin x}{\cos x} \qquad \sin^2 x + \cos^2 x = 1$$

We use trigonometric identities to simplify more complex expressions. A question would normally ask you to "Prove that…" a fact about sin, cos or tan is true or to "simplify" an expression involving sin, cos or tan.

There are not really any set rules for how to do these questions – instead you have to use your mathematical ability to choose the correct rules of algebra to find the answer. However the following tips are a good starting point:

Tip 1 – [illegible] ever "do stuff" to the more complicated expression (often the one on the left-hand side). Leave the simpler expression alone.

Tip 2 – you only get marks for knowing and using the two formulae in the grey box above. So a good bit of advice is to look at the more complicated expression, and:

a) If you see ' $\tan x$ ' on the left hand side, replace it with $\frac{\sin x}{\cos x}$

b) If you see ' $\frac{\sin x}{\cos x}$ ' on the left hand side, replace it with $\tan x$

c) If you see ' 1 ' on the left hand side, replace it with $\sin^2 x + \cos^2 x$

d) If you see ' $\sin^2 x + \cos^2 x$ ' on the left hand side, replace it with 1

<u>Example 1</u>

Prove that $\mathbf{\frac{1 - \cos^2 x}{3\sin^2 x} = \frac{1}{3}}$

Solution

Using Tip 1: The left-hand side is more complicated, and the right-hand side is simpler. This means that we will "do stuff" to the left-hand side $\frac{1 - \cos^2 x}{3\sin^2 x}$.

Using Tip 2: Using tip 2c above, we replace '1' with $\sin^2 x + \cos^2 x$:

$$\frac{1-\cos^2 x}{3\sin^2 x} = \frac{\sin^2 x + \cos^2 x - \cos^2 x}{3\sin^2 x}$$

We can now do some simplifying

$$\frac{\sin^2 x + \cancel{\cos^2 x} - \cancel{\cos^2 x}}{3\sin^2 x} = \frac{\sin^2 x}{3\sin^2 x} = \frac{{}^1\cancel{\sin^2 x}}{3\cancel{\sin^2 x}} = \frac{1}{3}$$, which is what we wanted

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<u>Example 2</u>

Simplify $5\sin^2 x + 5\cos^2 x$

Solution

Tip 1 isn't relevant here as there is only one expression.
Using **Tip 2**, none of those four expressions appear on the left-hand side exactly. However if we spot there is a common factor of 5 and <u>factorise it first</u>, we can use tip 2d:

$$\begin{aligned} 5\sin^2 x + 5\cos^2 x &= 5(\sin^2 x + \cos^2 x) \\ &= 5(1) \\ &= 5 \end{aligned}$$

. **Answer:** <u>5</u>

Exercise 1

1. Simplify

a) $3\cos^2 x + 3\sin^2 x$ **(b)** $1 - \cos^2 x$ **(c)** $\cos A \tan A$

d) $5 - 5\sin^2 B^o$ **(e)** $\dfrac{4\sin a^o}{4\cos a^o}$ **(f)** $\dfrac{4\tan x^o}{2\cos x^o}$

g) $\dfrac{(1-\sin^2 x)}{2\cos x}$ **(h)** $\dfrac{8-8\cos^2 x}{2\sin x}$ **(i)** $\dfrac{3\sin x\cos x}{6\tan x}$

j) $4\sin^2 A + 3\cos^2 A - 3$ **(k)** $4\cos^2 B - 2\sin^2 B + 2$

l) $(\cos x + \sin x)^2 - 2\sin x\cos x$ **(m)** $\tan^2 a(1 - \sin^2 a)$

2. Prove that

a) $2\cos^2 A + 3\sin^2 A = 3 - \cos^2 A$ **(b)** $\frac{1}{\tan x} + \tan x = \frac{1}{\sin x \cos x}$

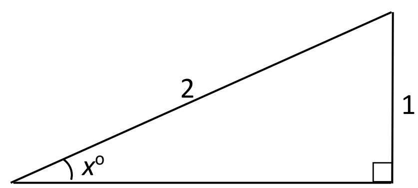
c) $(2\cos B + 3\sin B)^2 + (3\cos B - 2\sin B)^2 = 13$

d) $(1 + \sin x)(1 - \sin x) = \cos^2 x$ **(e)** $\sin \theta . \tan \theta = \frac{1 - \cos^2 \theta}{\cos \theta}$

3. Consider the diagram opposite :

a) Write down the **exact** values of .....

$\sin x^\circ$, $\cos x^\circ$ **and** $\tan x^\circ$.



b) Prove that $\sin^2 x^\circ + \cos^2 x^\circ = 1$.

c) Show that $\frac{\sin x^\circ}{\cos x^\circ} = \tan x^\circ$.

4. **a)** If $\cos a^\circ = \frac{1}{\sqrt{5}}$, find the **exact** values of $\sin a^\circ$ **and** $\tan a^\circ$ where $0 < a < 90$.

b) Prove that $\cos^2 a^\circ = 1 - \sin^2 a^\circ$.

c) Show that $\frac{\sin^2 a^\circ}{\cos^2 a^\circ} = \tan^2 a^\circ$.

d) Show that $2(3\sin a^\circ + 4\cos a^\circ) = 4\sqrt{5}$.

5. Prove that

a) $3\cos^2 a + 3\sin^2 a = 3$

b) $(\cos x + \sin x)^2 = 1 + 2\sin x \cos x$

c) $(\cos x + \sin x)(\cos x - \sin x) = 2\cos^2 x - 1$

d) $\dfrac{\sin x}{\cos x} + \dfrac{\cos x}{\sin x} = \dfrac{1}{\cos x \sin x}$

e) $\tan^2 p - \tan^2 p \sin^2 p = \sin^2 p$

f) $\cos^4 x - \sin^4 x = 2\cos^2 x - 1$

g) $3\sin^2 \theta + 2\cos^2 \theta = 2 + \sin^2 \theta$

6. Show that $(2\cos x + 5\sin x)^2 + (5\cos x - 2\sin x)^2 = 29$

7. Given that $p = \cos\theta + \sin\theta$ and $q = \cos\theta - \sin\theta$, prove that :

a) $pq = 2\cos^2 \theta - 1$ **(b)** $\sin^2 \theta = \frac{1}{2}(1 - pq)$

Answers

Exercise 1

1. **a)** $3(\cos^2 x + \sin^2 x) = 3 \times 1 = 3$ **(b)** $\sin^2 x$ **(c)** $\cancel{\cos A} \times \dfrac{\sin A}{\cancel{\cos A}} = \sin A$

d) $5(1 - \sin^2 B) = 5\cos^2 B$ **(e)** $\dfrac{4\sin a^o}{4\cos a^o} = \dfrac{\sin a^\circ}{\cos a^\circ} = \tan a^\circ$

f) $\dfrac{4\tan x^o}{2\cos x^o} = \dfrac{2\tan x^\circ}{\cos x^\circ} = \dfrac{2\dfrac{\sin x^\circ}{\cos x^\circ}}{\cos x} = 2\dfrac{\sin x^\circ}{\cos x^\circ} \times \dfrac{1}{\cos x^\circ} = \dfrac{2\sin x^\circ}{\cos^2 x}$

g) $\dfrac{(1-\sin^2 x)}{2\cos x} = \dfrac{\cos^2 x}{2\cos x} = \dfrac{\cos x}{2}$ **(h)** $\dfrac{8-8\cos^2 x}{2\sin x} = \dfrac{8(1-\cos^2 x)}{2\sin x} = \dfrac{8\sin^2 x}{2\sin x} = 4\sin x$

i) $\dfrac{3\sin x\cos x}{6\tan x} = \dfrac{3\sin x\cos x}{6\dfrac{\sin x}{\cos x}} = 3\sin x\cos x \times \dfrac{\cos x}{6\sin x} = \dfrac{\cos^2 x}{2}$

j) $4\sin^2 A + 3\cos^2 A - 3 = 3(\sin^2 A + \cos^2 A) + \sin^2 A - 3 = 3 + \sin^2 A - 3 = \sin^2 A$

k) $4\cos^2 B - 2\sin^2 B + 2 = 4\cos^2 B - 2(1-\cos^2 B) + 2 = 4\cos^2 B - 2 + \cos^2 B - 2 = 6\cos^2 B$

l) $(\cos x + \sin x)^2 - 2\sin x\cos x = \cos^2 x + 2\cos x\sin x + \sin^2 x - 2\sin x\cos x = \cos^2 x + \sin^2 x = 1$

m) $\tan^2 a(1 - \sin^2 a) = \dfrac{\sin^2\alpha}{\cos^2\alpha} \times \cos^2\alpha = \sin^2\alpha$

2. **a)** $3(\cos^2 A + \sin^2 A) - \cos^2 A \;=\; 3(1) - \cos^2 A = 3 - \cos^2 A$

b) $\dfrac{1}{\tan x} + \tan x = \dfrac{\cos x}{\sin x} + \dfrac{\sin x}{\cos x} = \dfrac{\cos^2 x + \sin^2 x}{\sin x\cos x} = \dfrac{1}{\sin x\cos x}$

c) $(2\cos B + 3\sin B)^2 + (3\cos B - 2\sin B)^2 =$

$4\cos^2 B + 12\cos B\sin B + 9\sin^2 B + 9\cos^2 B - 12\cos B\sin B + 4\sin^2 B$

$= 13\cos^2 B + 13\sin^2 B = 13(\cos^2 B + \sin^2 B) = 13$

d) $(1 + \sin x)(1 - \sin x) = 1 - \sin^2 x = \cos^2 x$

e) $\sin\theta.\tan\theta = \sin\theta \times \dfrac{\sin\theta}{\cos\theta} = \dfrac{\sin^2\theta}{\cos\theta} = \dfrac{1-\cos^2\theta}{\cos\theta}$

3. **a)** $\sin x^\circ = \frac{1}{2}$; $\cos x^\circ = \frac{\sqrt{3}}{2}$; $\tan x^\circ = \frac{1}{\sqrt{3}}$.

b) $\sin^2 x^\circ + \cos^2 x^\circ = (\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2 = \frac{1}{4} + \frac{3}{4} = 1$.

c) $\frac{\sin x^\circ}{\cos x^\circ} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{2} \times \frac{2}{\sqrt{3}} = \frac{1}{\sqrt{3}} = \tan x^\circ$.

4. **a)** $\sin a^\circ = \frac{2}{\sqrt{5}}$**;** $\tan a^\circ = 2$

b) $\cos^2 a^\circ = (\frac{1}{\sqrt{5}})^2 = \frac{1}{5}; 1 - (\frac{2}{\sqrt{5}})^2 = 1 - \frac{4}{5} = \frac{1}{5} = \cos^2 x$

c) $\frac{(\frac{2}{\sqrt{5}})^2}{(\frac{1}{\sqrt{5}})^2} = \frac{\frac{4}{5}}{\frac{1}{5}} = \frac{4}{5} \times \frac{5}{1} = 4;\ \tan^2 a^\circ = 2^2 = 4$.

d) $2[3(\frac{2}{\sqrt{5}}) + 4(\frac{1}{\sqrt{5}})] = 2(\frac{6}{\sqrt{5}} + \frac{4}{\sqrt{5}}) = \frac{20}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}} = \frac{20\sqrt{5}}{5} = 4\sqrt{5}$.

5. **a)** $3\cos^2 a + 3\sin^2 a = 3(\cos^2 a + \sin^2 a) = 3$

b) $(\cos x + \sin x)^2 = \cos^2 x + 2\sin x \cos x + \sin^2 x = 1 + 2\sin x \cos x$

c) $(\cos x + \sin x)(\cos x - \sin x) = \cos^2 x - \sin^2 x = \cos^2 x - (1 - \cos^2 x)$

$= \cos^2 x - 1 + \cos^2 x - 2\cos^2 x - 1$

d) $\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} = \frac{\sin^2 x + \cos^2 x}{\cos x \sin x} = \frac{1}{\cos x \sin x}$

e) $\tan^2 p - \tan^2 p \sin^2 p = \tan^2 p(1 - \sin^2 p) = \frac{\sin^2 p}{\cos^2 p} \times \cos^2 p = \sin^2 p$

f) $\cos^4 x - \sin^4 x = (\cos^2 x - \sin^2 x)(\cos^2 x + \sin^2 x)$

$= \cos^2 x - (1 - \cos^2 x) = \cos^2 x - 1 + \cos^2 x = 2\cos^2 x - 1$

g) $3\sin^2\theta + 2\cos^2\theta = 2(\sin^2\theta + \cos^2\theta) + \sin^2\theta = 2 + \sin^2\theta$

h) $\tan\alpha + \dfrac{1}{\tan\alpha} = \dfrac{\sin\alpha}{\cos\alpha} + \dfrac{\cos\alpha}{\sin\alpha} = \dfrac{\sin^2\alpha + \cos^2\alpha}{\sin\alpha\cos\alpha} = \dfrac{1}{\sin\alpha\cos\alpha}$

6.

$$(2\cos x + 5\sin x)^2 + (5\cos x - 2\sin x)^2$$
$$= 4\cos^2 x + 20\sin x\cos x + 25\sin^2 x + 25\cos^2 x - 20\sin x\cos x + 4\sin^2 x$$
$$= 29\cos^2 x + 29\sin^2 x = 29(\cos^2 x + \sin^2 x) = 29$$

7. **a)**

$$pq = (\cos\theta + \sin\theta)(\cos\theta - \sin\theta) = \cos^2\theta - \sin^2\theta$$
$$= \cos^2\theta - (1 - \cos^2\theta) = 2\cos^2\theta - 1$$

b) $\frac{1}{2}(1 - pq) = \frac{1}{2}[1 - (2\cos^2\theta - 1)] = \frac{1}{2}(2 - 2\cos^2\theta) = 1 - \cos^2\theta = \sin^2\theta$