5. Recurrence Relations

1 a) $U_{n+1} = 1.012U_n - 500$ $\qquad U_0 = 3000$

b) $U_1 = 1.012 \times 3000 - 500 = 2536$

$U_2 = 1.012 \times 2536 - 500 = 2066.432$

$U_3 = 1.012 \times 2066.432 - 500 = £1591.23$

c) $U_4 = 1110.32$

$U_5 = 623.64$

$U_6 = 131.12$

Payment 7 will pay off loan

2. a) $U_{n+1} = 1.045U_n - 1800$ $\qquad U_0 = 50,000$

b) $U_1 = 1.045 \times 50000 - 1800 = 50450$

$U_2 = 1.045 \times 50450 - 1800 = 50920.25$

$U_3 = 51411.66125$

$U_4 = 51925.18601$

$U_5 = £52461.82$

3. a) $U_{n+1} = 0.89U_n + 3$ $\qquad U_0 = 35$

b) $U_1 = 0.89 \times 35 + 3 = 34.15$

$U_2 = 0.89 \times 34.15 + 3 = 33.3935$

c) $U_3 = 0.89 \times 33.3935 + 3 = 32.7202$

This is the pressure at end of month 3 when 3psi is added. $32.7202 - 3 = 29.7202$ meaning the tyre fell under required pressure during this month.

4. a) $U_{n+1} = 0.2U_n + 4 \qquad U_0 = 3$

$U_1 = 0.2 \times 3 + 4 = 4.6$

$U_2 = 0.2 \times 4.6 + 4 = \underline{\underline{4.92}}$

b) $U_{n+1} = 0.1U_n + 5 \qquad U_0 = 7$

$U_1 = 0.1 \times 7 + 5 = 5.7$

$U_2 = 0.1 \times 5.7 + 5 = 5.57$

$U_3 = 0.1 \times 5.57 + 5 = 5.557$

$\Rightarrow \underline{\underline{5.56}}$

c) $U_{n+1} = -0.5U_n + 20 \qquad U_0 = 16$

$U_1 = -0.5 \times 16 + 20 = 12$

$U_2 = -0.5 \times 12 + 20 = 14$

$U_3 = -0.5 \times 14 + 20 = 13$

$U_4 = -0.5 \times 13 + 20 = \underline{\underline{13.5}}$

d) $U_{n+1} = -U_n - 7 \qquad U_0 = 1$

$U_1 = -1 - 7 = -8$

$U_2 = -(-8) - 7 = 1$

$U_3 = -1 - 7 = \underline{\underline{-8}}$

e) $U_n = 0.9U_{n-1} + 450 \qquad U_0 = 2$

$U_1 = 0.9 \times 2 + 450 = 451.8$

$U_2 = 0.9 \times 451.8 + 450 = \underline{\underline{856.62}}$

5. a) $V_{n+1} = 1.2V_n - 8 \quad V_0 = 150$

$V_1 = 1.2 \times 150 - 8 = 172$

$V_2 = 1.2 \times 172 - 8 = 198.4$

$V_3 = 1.2 \times 198.4 - 8 = \underline{\underline{230.08}}$

b) $V_4 = 268.096$

$V_5 = 313.7152$

$V_6 = 368.45824$

$V_7 = 434.1498888$

Value of $n > 400$ is $\underline{\underline{n = 7}}$

6. a) $U_{n+1} = 2U_n + 3 \quad U_0 = 3$

$U_1 = 2 \times 3 + 3 = 9$

$U_2 = 2 \times 9 + 3 = \underline{\underline{21}}$

no limit exists as $-1 < a < 1$

$\underline{\underline{a = 2}}$ here

b) $U_{n+1} = 0.7U_n + 12 \quad U_0 = 30$

$U_1 = 0.7 \times 30 + 12 = 33$

$U_2 = 0.7 \times 33 + 12 = 35.1$

$L = \frac{b}{1-a}$

$= \frac{12}{1-0.7}$

$= \underline{\underline{40}}$

7. a) $U_{n+1} = aU_n + b$

$U_{n+1} = 0.5U_n + 7 \quad U_0 = 30$

$U_1 = 0.5 \times 30 + 7 = 22$

$U_2 = 0.5 \times 22 + 7 = \underline{\underline{18}}$

b) Limit exists as $a = 0.5$ when $-1 < a < 1$

c) $L = \frac{b}{1-a} = \frac{7}{1-0.5} = \underline{\underline{14}}$

8. a) $u_{n+1} = k u_n - 5 \qquad u_0 = 0$

$$u_1 = k \times 0 - 5 \quad = -5$$

$$u_2 = (k \times -5) - 5 \quad = -5k - 5$$

as $u_2 = -7 \implies -5k - 5 = -7$

$$-5k = -2$$

$$\underline{\underline{k = \frac{2}{5}}}$$

b) i) limit exists as $k = \frac{2}{5} \quad \underline{\underline{-1 < \frac{2}{5} < 1}}$

ii) $L = \frac{b}{1-a} = \frac{-5}{1-\frac{2}{5}} = \underline{\underline{-\frac{25}{3}}} \quad (8.3\dot{3})$

9. a) $u_{n+1} = 0 \cdot 2 u_n + 4 \cdot 8$

$$v_{n+1} = k v_n + 4$$

$$L = \frac{b}{1-a} = \frac{4 \cdot 8}{1 - 0 \cdot 2} = \underline{\underline{6}}$$

b) As limits same $\quad \frac{4}{1-k} = 6$

$$4 = 6(1-k)$$

$$4 = 6 - 6k$$

$$6k = 2$$

$$\underline{\underline{k = \frac{1}{3}}}$$

10. $u_{n+1} = 0.2u_n + p$ $\quad u_0 = 1$

$v_{n+1} = 0.6v_n + q$ $\quad v_0 = 1$

$$\frac{p}{1-0.2} = \frac{q}{1-0.6}$$

$$\frac{p}{0.8} = \frac{q}{0.4}$$

$$p = \frac{0.8q}{0.4}$$

$$p = 2q$$

11. a) $u_{n+1} = au_n + b$ $\quad u_0 = 25$, $u_1 = 30$, $u_2 = 31$

$31 = 30a + b \;\dots\; ①$

$30 = 25a + b \;\dots\; ②$

$1 = 5a$

$a = \frac{1}{5}$

sub $a = \frac{1}{5}$ in ①

$31 = 30 \times \frac{1}{5} + b$

$31 = 6 + b$

$b = 25$

b) $L = \frac{b}{1-a}$

$= \frac{25}{1-\frac{1}{5}}$

$= \frac{125}{4} \quad (31\frac{1}{4})$

12. $U_{n+1} = mU_n + c$

$U_0 = 2 \quad U_1 = 4 \quad U_2 = 7$

$7 = 4m + c \quad \dots (1)$

$4 = 2m + c \quad \dots (2)$

$3 = 2m$

$\underline{\underline{m = \frac{3}{2}}}$

sub $m = \frac{3}{2}$ in (1)

$7 = 4 \times \frac{3}{2} + c$

$7 = 6 + c$

$\underline{\underline{c = 1}}$

13 a) $U_{n+1} = aU_n + b$

$U_2 = 190 \quad U_3 = 430 \quad U_4 = 910$

$910 = 430a + b \quad \dots (1)$

$430 = 190a + b \quad \dots (2)$

$480 = 240a$

$\underline{\underline{a = 2}}$

sub $a = 2$ in (1)

$910 = 430 \times 2 + b$

$\underline{\underline{b = 50}}$

b) $U_{n+1} = 2U_n + 50$

$U_2 = 190$

so

$190 = 2U_1 + 50$

$140 = 2U_1$

$\underline{\underline{U_1 = 70}}$

$U_1 = 70$

so

$70 = 2U_0 + 50$

$20 = 2U_0$

$\underline{\underline{U_0 = 10}}$

14. a) $U_{n+1} = 0 \cdot 3U_n + 220$

For day 1 though, the $+220$ hasn't been given a second time, so we would use

$U_{n+1} = 0 \cdot 3U_n$ when $U_n = 220$

$0 \cdot 3 \times 220 = \underline{\underline{66\text{mg}}}$

b) $U_{n+1} = 0 \cdot 3U_n + 220$

$a = 0 \cdot 3 \qquad b = 220$

c) Check limit for long term

$L = \frac{b}{1-a} = \frac{220}{1-0 \cdot 3} = 314 \cdot 3\text{mg}$

As $314 \cdot 3 < 350$ drug is safe