

7. Polynomials

1. a) $(6x^2 - 23x - 11) \div (x+2)$

$$\begin{array}{r|rrrr} -2 & 6 & -23 & -11 & \\ & & -12 & 70 & \\ \hline & 6 & -35 & 59 & \end{array}$$

$(6x - 35)$ rem 59

b) $(x^3 + 6x^2 - 3x - 9) \div (x-5)$

$$\begin{array}{r|rrrr} 5 & 1 & 6 & -3 & -9 \\ & & 5 & 55 & 260 \\ \hline & 1 & 11 & 52 & 251 \end{array}$$

$(x^2 + 11x + 52)$ rem 251

c) $(2x^3 + 3x^2 - 9x + 10) \div (x+5)$

$$\begin{array}{r|rrrr} -5 & 2 & 3 & -9 & 10 \\ & & -10 & 35 & -130 \\ \hline & 2 & -7 & 26 & -120 \end{array}$$

$(2x^2 - 7x + 26)$ rem -120

d) $(x^4 - 2x^2 + 5) \div (x-1)$

$$\begin{array}{r|rrrrr} 1 & 1 & 0 & -2 & 0 & 5 \\ & & 1 & 1 & -1 & -1 \\ \hline & 1 & 1 & -1 & -1 & 4 \end{array}$$

$(x^3 + x^2 - x - 1)$ rem 4

2 a) $(4x^2 + 6x - 2) \div (2x - 1)$

$$\begin{array}{r|rrr} \frac{1}{2} & 4 & 6 & -2 \\ & & 2 & 4 \\ \hline & 4 & 8 & 2 \end{array}$$

$$\Rightarrow (4x + 8) \times (x - \frac{1}{2}) + 2$$

$$= (4x^2 + 6x - 2)$$

so $2(2x + 4)(x - \frac{1}{2}) + 2$

$$(2x + 4)(2x - 1) + 2$$

$\uparrow$ $\uparrow$
 quotient remainder

b) $(2x^3 + x^2 + 4x + 4) \div (2x - 3)$

$$\begin{array}{r|rrrr} \frac{3}{2} & 2 & 1 & 4 & 4 \\ & & 3 & 6 & 15 \\ \hline & 2 & 4 & 10 & 19 \end{array}$$

$$\Rightarrow (2x^2 + 4x + 10) \times (x - \frac{3}{2}) + 19$$

$$= (2x^3 + x^2 + 4x + 4)$$

so $2(x^2 + 2x + 5)(x - \frac{3}{2}) + 19$

$$(x^2 + 2x + 5)(2x - 3) + 19$$

$\uparrow$ $\uparrow$
 quotient remainder

$$3. a) \begin{array}{r|rrrr} -4 & 1 & 8 & 11 & -20 \\ & & -4 & -16 & 20 \\ \hline & 1 & 4 & -5 & 0 \end{array}$$

As rem = 0 $(x+4)$ is factor

$$(x+4)(x^2+4x-5)$$

$$(x+4)(x-1)(x+5)$$

$$b) \begin{array}{r|rrrr} -1 & 2 & -3 & -3 & 2 \\ & & -2 & 5 & -2 \\ \hline & 2 & -5 & 2 & 0 \end{array}$$

As rem = 0 $(x+1)$ is a factor

$$(x+1)(2x^2-5x+2)$$

$$(x+1)(2x-1)(x-2)$$

$$c) \begin{array}{r|rrrr} 2 & 1 & -5 & 2 & 8 \\ & & 2 & -6 & -8 \\ \hline & 1 & -3 & -4 & 0 \end{array}$$

As rem = 0 $(x-2)$ is a factor

$$(x-2)(x^2-3x-4)$$

$$(x-2)(x+1)(x-4)$$

$$4. a) \begin{array}{r|rrrr} 2 & 1 & 0 & -7 & 6 \\ & & 2 & 4 & -6 \\ \hline & 1 & 2 & -3 & 0 \end{array}$$

$$(x-2)(x^2+2x-3)$$

$$(x-2)(x-1)(x+3)$$

$$b) \begin{array}{r|rrrr} 1 & 2 & 3 & -2 & -3 \\ & & 2 & 5 & 3 \\ \hline & 2 & 5 & 3 & 0 \end{array}$$

$$(x-1)(2x^2+5x+3)$$

$$(x-1)(2x+3)(x+1)$$

$$c) \begin{array}{r|rrrr} 3 & 2 & -1 & -13 & -6 \\ & & 6 & 15 & 6 \\ \hline & 2 & 5 & 2 & 0 \end{array}$$

$$(x-3)(2x^2+5x+2)$$

$$(x-3)(2x+1)(x+2)$$

$$d) \begin{array}{r|rrrr} 1 & 3 & 8 & -5 & -6 \\ & & 3 & 11 & 6 \\ \hline & 3 & 11 & 6 & 0 \end{array}$$

$$(x-1)(3x^2+11x+6)$$

$$(x-1)(3x+2)(x+3)$$

$$e) \begin{array}{r|rrrr} 1 & 1 & 5 & 5 & -5 & -6 \\ & & 1 & 6 & 11 & 6 \\ \hline & 1 & 6 & 11 & 6 & 0 \end{array}$$

$$(x-1)(x^3+6x^2+11x+6)$$

need to use synthetic division again.

$$\begin{array}{r|rrrr} -2 & 1 & 6 & 11 & 6 \\ & & -2 & -8 & -6 \\ \hline & 1 & 4 & 3 & 0 \end{array}$$

$$(x-1)(x+2)(x^2+4x+3)$$

$$(x-1)(x+2)(x+3)(x+1)$$

$$f) \begin{array}{r|rrrrr} -1 & 2 & 6 & 6 & 2 & 0 \\ & & -2 & -4 & -2 & 0 \\ \hline & 2 & 4 & 2 & 0 & 0 \end{array}$$

$$(x+1)(2x^3+4x^2+2x)$$

$$\Rightarrow (x+1)(x)(2x^2+4x+2)$$

$$\Rightarrow (x+1)(x)(2)(x^2+2x+1)$$

$$\Rightarrow \underline{\underline{2x(x+1)(x+1)(x+1)}}$$

$$5. a) \begin{array}{l|rrrr} -4 & 1 & -5 & -p & 80 \\ & -4 & 36 & 4p & -144 \\ \hline & 1 & -9 & 36-p & 4p-64=0 \end{array}$$

$$\begin{aligned} 4p-64 &= 0 \\ 4p &= 64 \\ \underline{\underline{p &= 16}} \end{aligned}$$

$$b) \begin{array}{l|rrrr} 4 & 1 & -9 & p & -28 \\ & 4 & -20 & 4p & -80 \\ \hline & 1 & -5 & p-20 & 4p-108=0 \end{array}$$

$$\begin{aligned} 4p-108 &= 0 \\ 4p &= 108 \\ \underline{\underline{p &= 27}} \end{aligned}$$

$$c) \begin{array}{l|rrrrr} -5 & 3 & 15 & -p & -9 & 5 \\ & -15 & 0 & 5p & -25p+45 \\ \hline & 3 & 0 & -p & 5p-9 & -25p+50=0 \end{array}$$

$$\begin{aligned} -25p+50 &= 0 \\ 25p &= 50 \\ \underline{\underline{p &= 2}} \end{aligned}$$

$$6 a) \begin{array}{l|rrrr} 2 & 2 & p & q & -6 \\ & 4 & 2p+8 & 4p+2q+16 \\ \hline & 2 & p+4 & 2p+q+8 & \underline{4p+2q+10=0} \end{array}$$

$$\begin{array}{l|rrrr} 3 & 2 & p & q & -6 \\ & 6 & 3p+18 & 9p+3q+54 \\ \hline & 2 & p+6 & 3p+q+18 & \underline{9p+3q+48=0} \end{array}$$

$$\begin{aligned} 4p+2q &= -10 \dots \textcircled{1} \times -3 \\ 9p+3q &= -48 \dots \textcircled{2} \times 2 \end{aligned}$$

Elim q

$$\begin{aligned} -12p-6q &= 30 \\ 18p+6q &= -96 \\ \hline 6p &= -66 \end{aligned}$$

$$\underline{\underline{p = -11}}$$

$$\begin{aligned} \text{Sub } p &= -11 \text{ in } \textcircled{1} \\ (4 \times -11) + 2q &= -10 \end{aligned}$$

$$2q = -10 + 44$$

$$2q = 34$$

$$\underline{\underline{q = 17}}$$

$$b) \quad \begin{array}{c|cccc} 3 & 2 & p & q & -15 \\ \hline & 6 & 3p+18 & 9p+3q+54 & \\ \hline & 2 & p+6 & 3p+q+18 & 9p+3q+39=0 \end{array}$$

$$\begin{array}{c|cccc} -5 & 2 & p & q & -15 \\ \hline & -10 & -5p+50 & 25p-5q-250 & \\ \hline & 2 & p-10 & -5p+q+50 & 25p-5q-265=0 \end{array}$$

$$9p+3q = -39 \quad \times 5$$

$$25p-5q = 265 \quad \times 3$$

elim q

$$45p+15q = -195$$

$$75p-15q = 795$$

$$\hline 120p = 600$$

$$p = \underline{\underline{5}}$$

Sub $p = 5$ in ①

$$(9 \times 5) + 3q = -39$$

$$3q = -39 - 45$$

$$3q = -84$$

$$\underline{\underline{q = -28}}$$

$$c) \quad \begin{array}{c|cccc} -2 & p & 0 & q & 2 \\ \hline & -2p & 4p & -8p-2q & \\ \hline & p & -2p & 4p+q & -8p-2q+2=0 \end{array}$$

$$\begin{array}{c|cccc} 1 & p & 0 & q & 2 \\ \hline & p & p & p+q & \\ \hline & p & p & p+q & p+q+2=0 \end{array}$$

$$-8p-2q = -2 \quad \dots \text{①}$$

$$p+q = -2 \quad \dots \text{②} \times 2$$

elim q

$$-8p-2q = -2$$

$$2p+2q = -4$$

$$\hline -6p = -6$$

$$\underline{\underline{p = 1}}$$

Sub $p = 1$ in ②

$$1+q = -2$$

$$q = -2-1$$

$$\underline{\underline{q = -3}}$$

$$\begin{array}{r|rrrr}
 2 & 2 & a & b & 4 \\
 & 4 & 2a+8 & 4a+2b+16 & \\
 \hline
 & 2 & a+4 & 2a+b+8 & 4a+2b+20=0
 \end{array}$$

$$\begin{array}{r|rrrr}
 5 & 2 & a & b & 4 \\
 & 10 & 5a+50 & 25a+5b+250 & \\
 \hline
 & 2 & a+10 & 5a+b+50 & 25a+5b+254=54
 \end{array}$$

$$\Rightarrow \underline{\underline{25a+5b+200=0}}$$

$$\begin{aligned}
 4a+2b &= -20 \dots \textcircled{1} \times 5 \\
 25a+5b &= -200 \dots \textcircled{2} \times 2
 \end{aligned}$$

Elim b

$$\begin{array}{r}
 -20a - 10b = 100 \\
 50a + 10b = -400 \\
 \hline
 30a = -300 \\
 \underline{\underline{a = -10}}
 \end{array}$$

Sub $a = -10$ in $\textcircled{1}$

$$\begin{aligned}
 (4 \times -10) + 2b &= -20 \\
 2b &= -20 + 40 \\
 \underline{\underline{b &= 10}}
 \end{aligned}$$

$$\begin{array}{r|rrrr}
 1 & 1 & -1 & 1 & a & b \\
 & 1 & 0 & 1 & a+1 & \\
 \hline
 & 1 & 0 & 1 & a+1 & \underline{\underline{a+b+1=0}}
 \end{array}$$

$$\begin{array}{r|rrrr}
 2 & 1 & -1 & 1 & a & b \\
 & 2 & 2 & 6 & 2a+12 & \\
 \hline
 & 1 & 1 & 3 & a+6 & 2a+b+12=11
 \end{array}$$

$$\Rightarrow \underline{\underline{2a+b=-1}}$$

$$a+b=-1 \dots \textcircled{1} \times -1$$

$$2a+b=-1 \dots \textcircled{2}$$

Elim b

$$\begin{array}{r}
 -a+b=1 \\
 2a+b=-1 \\
 \hline
 a=0
 \end{array}$$

Sub $a=0$ in $\textcircled{1}$

$$\begin{aligned}
 0+b &= -1 \\
 \underline{\underline{b &= -1}}
 \end{aligned}$$

9.
$$\begin{array}{r|rrrr} 2 & 10 & -27 & a & b \\ & 20 & -14 & 2a-28 & \\ \hline & 10 & -7 & a-14 & 2a+b-28 = -45 \\ & & & & \Rightarrow \underline{\underline{2a+b = -17}} \end{array}$$

$$\begin{array}{r|rrrr} -1 & 10 & -27 & a & b \\ & -10 & 37 & -a-37 & \\ \hline & 10 & -37 & a+37 & -a+b-37 = -24 \\ & & & & \Rightarrow \underline{\underline{-a+b = 13}} \end{array}$$

$$2a+b = -17 \dots (1)$$

$$-a+b = 13 \dots (2) \quad \times -1$$

Elim b

$$2a+b = -17$$

$$a-b = -13$$

$$\hline 3a = -30$$

$$\underline{\underline{a = -10}}$$

Sub $a = -10$ in (2)

$$-(-10)+b = 13$$

$$b = 13-10$$

$$\underline{\underline{b = 3}}$$

10 a) $y = k(x-1)(x-4)$ at $(0, 8)$
 $8 = k(0-1)(0-4)$
 $8 = k \times (-1)(-4)$
 $8 = 4k$
 $\underline{k = 2} \Rightarrow \underline{y = 2(x-1)(x-4)}$

b) $y = k(x+1)(x-5)$ at $(0, 10)$
 $10 = k(0+1)(0-5)$
 $10 = -5k$
 $\underline{k = -2} \Rightarrow \underline{y = -2(x+1)(x-5)}$

c) $y = k(x+4)(x+1)(x-4)$ at $(0, -32)$
 $-32 = k(0+4)(0+1)(0-4)$
 $-32 = -16k$
 $\underline{k = 2} \Rightarrow \underline{y = 2(x+4)(x+1)(x-4)}$

d) $y = k(x+1)(x-3)(x-3)$ at $(0, 27)$
(repeat root)
 $27 = k(0+1)(0-3)(0-3)$
 $27 = 9k$
 $\underline{k = 3} \Rightarrow \underline{y = 3(x+1)(x-3)(x-3)}$

e) $y = k(x+3)(x-1)(x-2)$ at $(0, 12)$
 $12 = k(0+3)(0-1)(0-2)$
 $12 = 6k$
 $\underline{k = 2} \Rightarrow \underline{y = 2(x+3)(x-1)(x-2)}$

f) $y = k(x+2)(x+2)(x-4)$ at $(0, 32)$
 $32 = k(0+2)(0+2)(0-4)$
 $32 = -16k$
 $\underline{k = -2} \Rightarrow \underline{y = -2(x+2)(x+2)(x-4)}$

g) $y = k(x+4)(x-8)(x-8)(x-10)$ at $(0, -640)$
 $-640 = k(0+4)(0-8)(0-8)(0-10)$
 $-640 = -2560k$
 $\underline{k = \frac{1}{4}} \Rightarrow \underline{y = \frac{1}{4}(x+4)(x-8)(x-8)(x-10)}$

h) $y = k(x+3)(x+3)(x-3)(x-3)$ at $(0, -27)$
 $-27 = k(0+3)(0+3)(0-3)(0-3)$
 $-27 = 81k$
 $\underline{k = -\frac{1}{3}} \Rightarrow \underline{y = -\frac{1}{3}(x+3)(x+3)(x-3)(x-3)}$

$$11a) x^3 + 2x^2 - 5 = 0$$

$$x=1 \quad 1^3 + 2 \times 1^2 - 5 = -2$$

$$x=1.5 \quad (1.5)^3 + (2 \times 1.5^2) - 5 = 2.875$$

$$\text{Try } x=1.25 \quad (1.25)^3 + (2 \times 1.25^2) - 5 = 0.078125 *$$

This is very close to zero so root is approximately at $x=1.25$ (2dp).

(If you do $x=1.24$ you get -0.018176 meaning root is somewhere between 1.24 and 1.25)

$$b) x^3 - x^2 - 2x + 1 = 0$$

$$x=1.5 \quad (1.5)^3 - (1.5)^2 - (2 \times 1.5) + 1 = -0.875$$

$$x=2 \quad (2)^3 - (2)^2 - (2 \times 2) + 1 = 1$$

$$\text{Try } x=1.75 \quad (1.75)^3 - (1.75)^2 - (2 \times 1.75) + 1 = -0.203125$$

$$\text{Try } x=1.80 \quad (1.80)^3 - (1.80)^2 - (2 \times 1.80) + 1 = \underline{\underline{-0.008}}$$

Approximate root is therefore $x=1.80$ (2dp)

12. a) $f(x) = x^3 + 1$ $g(x) = 2x + 1$ $h(x) = 17 - 11x$

$$\begin{aligned} g(f(x)) &= g(x^3 + 1) \\ &= 2(x^3 + 1) + 1 \\ &= 2x^3 + 2 + 1 \\ &= \underline{\underline{2x^3 + 3}} \end{aligned}$$

b) $g(f(x)) + x(h(x)) - 9 = 2x^3 - 11x^2 + 17x - 6$

$$\begin{aligned} \text{LHS} &= (2x^3 + 3) + x(17 - 11x) - 9 \\ &= 2x^3 + 3 + 17x - 11x^2 - 9 \\ &= 2x^3 - 11x^2 + 17x - 6 \\ &= \underline{\underline{\text{RHS}}} \end{aligned}$$

c) i)
$$\begin{array}{r|rrrr} 2 & 2 & -11 & 17 & -6 \\ & & 4 & 14 & 6 \\ \hline & 2 & -7 & 3 & 0 \end{array}$$
 As rem = 0 $(x-2)$ is a factor

ii) $(x-2)(2x^2 - 7x + 3)$
 $(x-2)(2x-1)(x-3)$

d) $g(f(x)) + xh(x) = 9$ is the same as
 $g(f(x)) + xh(x) - 9 = 0$ so to solve we use
 c) ii)

$$(x-2)(2x-1)(x-3) = 0$$

$$\underline{\underline{x=2}} \quad \underline{\underline{x=\frac{1}{2}}} \quad \underline{\underline{x=3}}$$