Wave Function

SPTA Mathematics – Higher Notes

- A Trig expression in the form $a\cos x° + b\sin x°$ can be written in the form: $k\cos(x \pm \alpha)°$ or $k\sin(x \pm \alpha)°$

  $k = \sqrt{a^2 + b^2}$

  $Tan\alpha° = \frac{k\sin\alpha°}{k\cos\alpha°}$

- You can convert these expressions by following the steps below:
  - Expand the expression $k\cos(x \pm \alpha)°$ or $k\sin(x \pm \alpha)°$ using one of:
    - $k\sin(A \pm B) = k\sin A\cos B \pm k\cos A\sin B$
    - $k\cos(A \pm B) = k\cos A\cos B \mp k\sin A\sin B$
  - Compare with the given expression to write down $k\cos\alpha°$ and $k\sin\alpha°$
  - Use the CAST diagram to find the quadrant for angle $\alpha°$. ← **Look for 2 ticks!!**
  - Use the 2 formulae above to calculate k and α.
  - State the expression in the form asked for.

Examples:

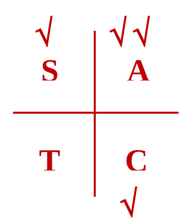
1. Write $5\cos x + 12\sin x$ in the form $k\cos(x - \alpha)°$ where $0 \leq \alpha \leq 360°$

$k\cos(x - \alpha)° = k\cos x\cos\alpha + k\sin x\sin\alpha$ ← **Step 1**

$5\cos x + 12\sin x$ ← **Step 2**

so $k\cos\alpha = 5$ and $k\sin\alpha = 12$ ← **Step 2**

Step 3



Hence α is in 1st quadrant

Step 4

$k = \sqrt{a^2 + b^2}$

$k = \sqrt{5^2 + 12^2}$

$k = \sqrt{25 + 144}$

$k = \sqrt{169}$

$k = 13$

$Tan\alpha = \frac{k\sin\alpha}{k\cos\alpha}$

$Tan\alpha = \frac{12}{5}$

$\alpha = \tan^{-1} 2.4$

$\alpha = 67.38°$

So $5\cos x + 12\sin x = 13\cos(x - 67.38)°$ ← **Step 5**

2. Write 5sin$x$ – 3cos$x$ in the form $k$sin(x − α) where $0 \le \alpha \le 2\pi$

$k\sin(x - \alpha) = k\sin x\cos\alpha - k\cos x\sin\alpha$

$\qquad\qquad\qquad 5\sin x - 3\cos x$

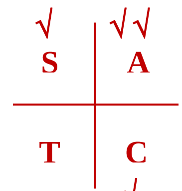
so $k\cos\alpha = 5$ and $k\sin\alpha = 3$

$k = \sqrt{5^2 + 3^2}$ $\qquad Tan\alpha = \frac{3}{5}$

$k = \sqrt{25 + 9}$ $\qquad \alpha = \tan^{-1}\frac{3}{5}$

$k = \sqrt{34}$ $\qquad \alpha = 30.96°$

$\alpha = 0.540$



Hence α is in 1st quadrant

So $5\sin x - 3\cos x = \sqrt{34}\sin(x - 0.54)$

Convert to RADIANS by multiplying by π and dividing by 180. Only exact value radians are left as a fraction of π. Alternatively change your calculator to radians before doing INV tan but remember to change it back after!!!

3. Write 4cos$x$ – 5sin$x$ in the form $k$sin(x – α)° where $0 \le \alpha \le 360°$

$k\sin(x + \alpha)° = k\sin x\cos\alpha + k\cos x\sin\alpha$

$\qquad\qquad\qquad 4\cos x - 5\sin x$

$\qquad\qquad\qquad -5\sin x + 4\cos x$

Rearrange to match the expansion.

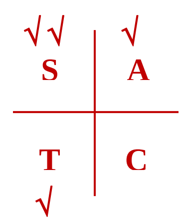
so $k\cos\alpha = -5$ and $k\sin\alpha = 4$

$k = \sqrt{(-5)^2 + 4^2}$ $\qquad Tan\alpha = \frac{4}{-5}$

$k = \sqrt{25 + 16}$ $\qquad \alpha = \tan^{-1}\left(-\frac{4}{5}\right)$

$k = \sqrt{41}$ $\qquad \alpha = 180 - 38.66°$

$\alpha = 141.34°$



Hence α is in 2nd quadrant

So $4\cos x - 5\sin x = \sqrt{41}\sin(x - 141.34)°$

Never put a negative Trig ratio into your calculator – use the CAST diagram!!

Sometimes there is a multiple x term such as $2x$ or $3x$, but this makes no difference to the process.

4. Write $-\sqrt{3}\cos 2x - 2\sin 2x$ in the form $k\cos(2x + \alpha)°$ where $0 \leq \alpha \leq 360°$

$k\cos(2x + \alpha)° = k\cos 2x\cos\alpha - k\sin 2x\sin\alpha$

$-2\sin 2x - \sqrt{3}\cos 2x$

$-\sqrt{3}\cos 2x - 2\sin 2x$

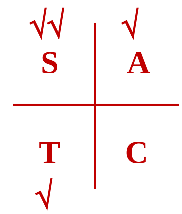
so $k\cos\alpha = -\sqrt{3}$ and $k\sin\alpha = 2$

$k = \sqrt{(-\sqrt{3})^2 + 2^2}$ $\qquad$ $Tan\alpha = \frac{2}{-\sqrt{3}}$

$k = \sqrt{3 + 4}$ $\qquad$ $\alpha = \tan^{-1}(-\frac{2}{\sqrt{3}})$

$k = \sqrt{7}$ $\qquad$ $\alpha = 180 - 49.11°$

$\alpha = 130.89°$



Hence α is in 3rd quadrant

So $-\sqrt{3}\cos 2x - 2\sin 2x = \sqrt{7}\sin(2x + 130.89)°$

Now attempt Exercise 1 from the Wave Function booklet.

Maximum and Minimum Values:

- Remember that $y = \sin x°$ and $y = \cos x°$ have a Maximum of 1 and a Minimum of -1.
- We can find the Max/Min values of expressions in the form $a\cos x° + b\sin x°$ by firstly converting them into a single trig expression in the form: $y = k\cos(x \pm \alpha)°$ or $y = k\sin(x \pm \alpha)°$.

Examples:

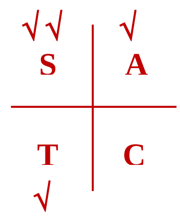
5. a) Write $4\sin x - \cos x$ in the form $k\cos(x - \alpha)°$ where $0 \leq \alpha \leq 360°$

$$k\cos(x-\alpha)° = k\cos x\cos\alpha + k\sin x\sin\alpha$$
$$-\cos x + 4\sin x$$

so $k\cos\alpha = -1$ and $k\sin\alpha = 4$

$$k = \sqrt{(-1)^2 + 4^2}$$
$$k = \sqrt{1+16}$$
$$k = \sqrt{17}$$

$$Tan\alpha = \frac{4}{-1}$$
$$\alpha = \tan^{-1}(-4)$$
$$\alpha = 180 - 75.96°$$
$$\alpha = 104.04°$$



Hence α is in 2nd quadrant

So $4\sin x - \cos x = \sqrt{17}\cos(x - 104.04)°$

b) State the maximum and minimum values and the corresponding values of x.

Max/Min values of $\cos x$ from the graph is 1 & -1 respectively

So Max $= \sqrt{17}$ and Min $= -\sqrt{17}$

Maximum occurs when: $\cos(x - 104.04)° = 1$

$$x - 104.04° = 0° \text{ or } 360°$$
$$x = 104.04° \text{ or } ~~464.04°~~$$
$$x = 104.04°$$

State the values from the Trig graphs

Minimum occurs when: $\cos(x - 104.04)° = -1$

$$x - 104.04° = 180°$$
$$x = 284.04°$$

Now attempt Exercise 2 from the Wave Function booklet.

Solving Equations:

- The wave function can be used to solve equations in the form acos(nx)° + bsin(nx)° by first converting it into a single trig expression in the form: $y = k\cos(x \pm \alpha)°$ or $y = k\sin(x \pm \alpha)°$.

Examples:

6. Solve $5\cos x + \sin x = 2$ where $0 \le \alpha \le 180°$

$$k\sin(x+\alpha)° = k\sin x\cos\alpha + k\cos x\sin\alpha$$

$$5\cos x + \sin x$$

$$\sin x + 5\cos x$$

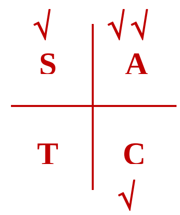
Choose an expansion with the same sign in the middle

so $k\cos\alpha = 1$ and $k\sin\alpha = 5$

$$k = \sqrt{5^2 + 1^2} \qquad Tan\alpha = \frac{5}{1}$$

$$k = \sqrt{25+1} \qquad \alpha = \tan^{-1} 5$$

$$k = \sqrt{26} \qquad \alpha = 78.69°$$



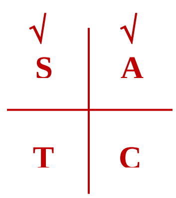
Hence α is in 1st quadrant

So $5\cos x + \sin x = \sqrt{26}\cos(x - 78.69)°$

So $5\cos x + \sin x = 2 \implies \sqrt{26}\sin(x + 78.69)° = 2$

$$\sin(x + 78.69)° = \frac{2}{\sqrt{26}}$$

2 solutions per x, use the CAST diagram to find the related angle.



$$x + 78.69° = \sin^{-1}\left(\frac{2}{\sqrt{26}}\right)$$

$$x + 78.69° = 23.09° \text{ or } 180 + 23.09°$$

$$x + 78.69° = 23.09° \text{ or } 203.09°$$

$$x = -55.6° \text{ or } 124.4°$$

$$x = \cancel{304.4°} \text{ or } 124.4°$$

$$x = 124.4°$$

Outwith range

7. Solve $2\cos 2x - 3\sin 2x = 1$ where $0 \le \alpha \le 2\pi$

$k\cos(2x + \alpha)° = k\cos 2x\cos\alpha - k\sin 2x\sin\alpha$

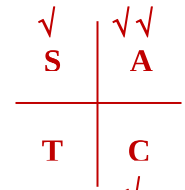
$2\cos 2x - 3\sin 2x$

so $k\cos\alpha = 2$ and $k\sin\alpha = 3$

$k = \sqrt{2^2 + 3^2}$ $\qquad Tan\alpha = \frac{3}{2}$

$k = \sqrt{4 + 9}$ $\qquad \alpha = \tan^{-1}(1.5)$

$k = \sqrt{13}$ $\qquad \alpha = 56.31°$



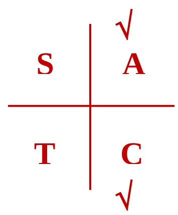
Hence α is in 1st quadrant

So $2\cos 2x - 3\sin 2x = \sqrt{13}\cos(2x + 56.31)°$

So $2\cos 2x - 3\sin 2x = \sqrt{13}\cos(2x + 0.984)°$

So $2\cos 2x - 3\sin 2x = 1 \implies \sqrt{13}\cos(2x + 56.31)° = 1$

$\cos(2x + 56.31)° = \frac{1}{\sqrt{13}}$



$2x + 56.31° = \cos^{-1}\left(\frac{1}{\sqrt{13}}\right)$

$2x + 56.31° = 73.9°$ or $360 - 73.9°$

$2x + 56.31° = 73.9°$ or $286.1°$ or $433.9°$ or $646.1°$

$2x = 17.59°, 229.79°, 377.59°, 589.79°$

$x = 8.80°, 114.90°, 188.80°, 294.90°$

$x = 0.154, 2.005, 3.30, 5.147$ radians

2 solutions per x, since 2x there are 4 solutions

Convert to RADIANS by multiplying by π and dividing by 180. Only exact value radians are left as a fraction of π. Alternatively change your calculator to radians before doing INV tan but remember to change it back after!!!

Now attempt Exercise 3 from the Wave Function booklet.

Sketching Graphs:

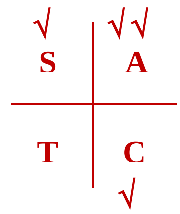
- You may also be asked to Draw the Graph of an equation in the form $y = a\cos(nx)° + b\sin(nx)°$ by first converting it into a single trig expression.
- We saw earlier in the course how to sketch Trig Graphs.

Examples:

8. Sketch the graph of the trig equation $y = \sqrt{3}\sin 2x + \cos 2x$ where $0 \leq \alpha \leq 360°$

$$k\sin(2x + \alpha)° = k\sin 2x\cos\alpha + k\cos 2x\sin\alpha$$
$$\sqrt{3}\sin 2x + \cos 2x$$

so $k\cos\alpha = \sqrt{3}$ and $k\sin\alpha = 1$



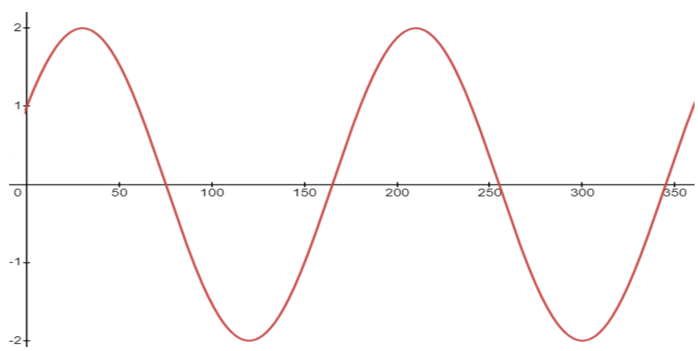
Hence α is in 1st quadrant

$$k = \sqrt{(\sqrt{3})^2 + 1^2}$$
$$k = \sqrt{3 + 1}$$
$$k = \sqrt{4} = 2$$

$$Tan\alpha = \frac{1}{\sqrt{3}}$$
$$\alpha = \tan^{-1}(\frac{1}{\sqrt{3}})$$
$$\alpha = 30°$$

So $\sqrt{3}\sin 2x + \cos 2x = 2\sin(2x + 30)°$

So the Graph of $y = 2\sin(2x + 30)°$



Now attempt Exercise 4 – 6 from the Wave Function booklet.