Vectors

SPTA Mathematics – Higher Notes

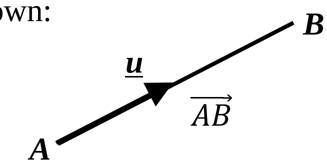
Definitions:

Scan here for the definition of a Vector

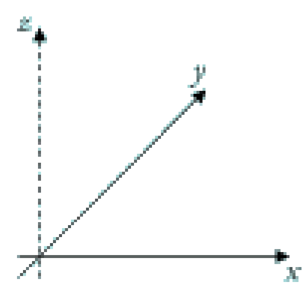
- MAGNITUDE - the SIZE of an object
- SCALAR - quantities or objects that have magnitude only,
  i.e. Length, Mass, Number of Sharks in the sea, Height of a wall, etc.
- VECTOR - quantities or objects that have magnitude **AND** direction,
  i.e. Weight, A plane flies 300km on a bearing of 145°
  Wind velocity is 15mph southeasterly.

Vectors:

- A vector can be named in 2 ways meaning the same thing:
  - A Bold, italicised and underlined letter, $\underline{\boldsymbol{u}}$ represents Vector $\underline{\boldsymbol{u}}$
  - 2 capital letters with an arrow above, $\overrightarrow{AB}$, represents the Directed Line Segment *AB*.
- A vector can also be represented graphically as shown:



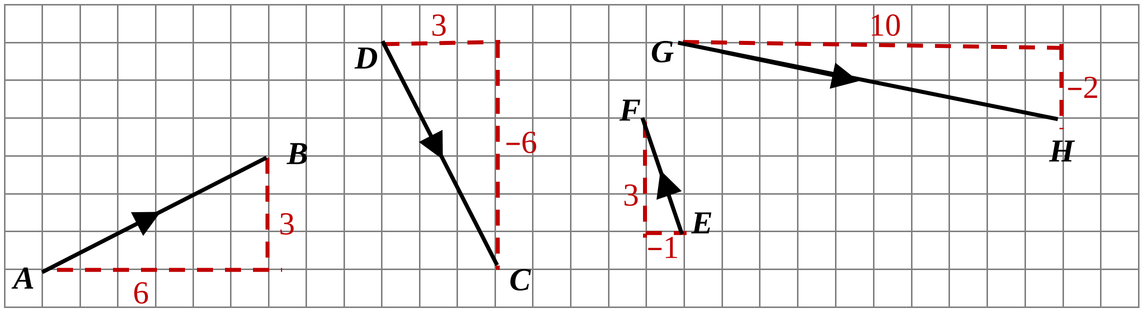
- A vector can be described using components, in:
  - 2D $\underline{\boldsymbol{u}} = \begin{pmatrix} x \\ y \end{pmatrix}$, e.g. $\underline{\boldsymbol{u}} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$ meaning 3 RIGHT and 4 DOWN.
  - 3D $\underline{\boldsymbol{u}} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, e.g. $\underline{\boldsymbol{u}} = \begin{pmatrix} -5 \\ 3 \\ 2 \end{pmatrix}$ meaning 5 LEFT, 3 IN & 2 UP.



- Vectors with the same components are equal vectors, regardless of their starting point, i.e.
  $\overrightarrow{AB} = \begin{pmatrix} -2 \\ 5 \\ 4 \end{pmatrix}, \overrightarrow{CD} = \begin{pmatrix} -2 \\ 5 \\ 4 \end{pmatrix}$ hence $\overrightarrow{AB} = \overrightarrow{CD}$
- Remember direction matters so $\overrightarrow{AB} \neq \overrightarrow{BA}$

Example:

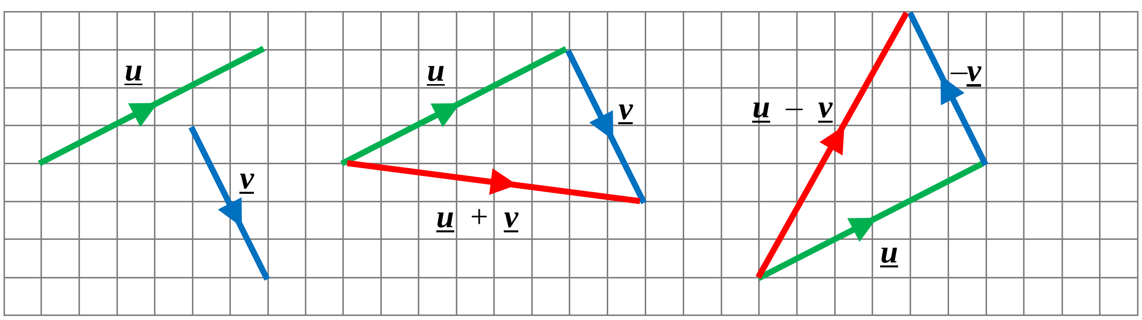
1. Write down the components of the Vectors below:



$\overrightarrow{AB} = \binom{6}{3}$ $\overrightarrow{DC} = \binom{3}{-6}$ $\overrightarrow{EF} = \binom{-1}{3}$ $\overrightarrow{GH} = \binom{10}{-2}$

Vector Arithmetic:

- When you add or subtract Vectors you produce another vector called the **RESULTANT VECTOR**.
- When subtracting Vectors you actually add the negative Vector, i.e. $\underline{\boldsymbol{u}} - \underline{\boldsymbol{v}} = \underline{\boldsymbol{u}} + (-\underline{\boldsymbol{v}})$
- Vectors are added/subtracted using the Nose to Tail method as shown:



- Their components can be added/subtracted as follows:

If $\underline{\boldsymbol{u}} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $\underline{\boldsymbol{v}} = \begin{pmatrix} d \\ e \\ f \end{pmatrix}$ then $\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}} = \begin{pmatrix} a+d \\ b+e \\ c+f \end{pmatrix}$ and $\underline{\boldsymbol{u}} - \underline{\boldsymbol{v}} = \begin{pmatrix} a-d \\ b-e \\ c-f \end{pmatrix}$

- Multiplying a Vector by a **Scalar** looks like this: If $\underline{\boldsymbol{u}} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ then $k\underline{\boldsymbol{u}} = \begin{pmatrix} kx \\ ky \\ kz \end{pmatrix}$

- You **cannot** multiply a Vector by another Vector.

Examples:

2. **a)** For the example above, state the components of the vectors $\underline{\boldsymbol{u}}$ and $\underline{\boldsymbol{v}}$.

Find the resultant vectors given by: **b)** $\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}}$ **(c)** $\underline{\boldsymbol{u}} - \underline{\boldsymbol{v}}$ **(d)** $3\underline{\boldsymbol{u}}$ **(e)** $3\underline{\boldsymbol{u}} + 5\underline{\boldsymbol{v}}$

a) $\underline{\boldsymbol{u}} = \begin{pmatrix} 6 \\ 3 \end{pmatrix}$ and $\underline{\boldsymbol{v}} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$

b) $\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}} = \begin{pmatrix} 6 \\ 3 \end{pmatrix} + \begin{pmatrix} 2 \\ -4 \end{pmatrix}$

$= \begin{pmatrix} 8 \\ -1 \end{pmatrix}$

(c) $\underline{\boldsymbol{u}} - \underline{\boldsymbol{v}} = \begin{pmatrix} 6 \\ 3 \end{pmatrix} - \begin{pmatrix} 2 \\ -4 \end{pmatrix}$

$= \begin{pmatrix} 4 \\ 5 \end{pmatrix}$

d) $3\underline{\boldsymbol{u}} = 3\begin{pmatrix} 6 \\ 3 \end{pmatrix}$

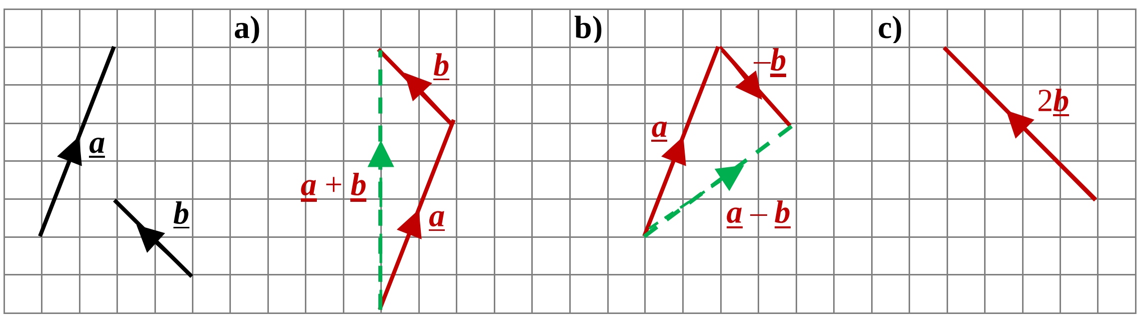
$= \begin{pmatrix} 18 \\ 9 \end{pmatrix}$

(e) $3\underline{\boldsymbol{u}} + 5\underline{\boldsymbol{v}} = 3\begin{pmatrix} 6 \\ 3 \end{pmatrix} + 5\begin{pmatrix} 2 \\ -4 \end{pmatrix}$

$= \begin{pmatrix} 18 \\ 9 \end{pmatrix} + \begin{pmatrix} 10 \\ -20 \end{pmatrix}$

$= \begin{pmatrix} 28 \\ -11 \end{pmatrix}$

3. For the Vectors below draw the resultant Vector: **a)** $\underline{\boldsymbol{a}} + \underline{\boldsymbol{b}}$ **(b)** $\underline{\boldsymbol{a}} - \underline{\boldsymbol{b}}$ **(c)** $2\underline{\boldsymbol{b}}$



4. State the components of the resultant Vectors when $\underline{\boldsymbol{m}} = \begin{pmatrix} 2 \\ -4 \\ 6 \end{pmatrix}$ and $\underline{\boldsymbol{n}} = \begin{pmatrix} -3 \\ 5 \\ -1 \end{pmatrix}$:

a) $2\underline{\boldsymbol{m}} - 3\underline{\boldsymbol{n}} = 2\begin{pmatrix} 2 \\ -4 \\ 6 \end{pmatrix} - 3\begin{pmatrix} -3 \\ 5 \\ -1 \end{pmatrix}$

$= \begin{pmatrix} 4 \\ -8 \\ 12 \end{pmatrix} - \begin{pmatrix} -9 \\ 15 \\ -3 \end{pmatrix}$

$= \begin{pmatrix} 13 \\ -23 \\ 15 \end{pmatrix}$

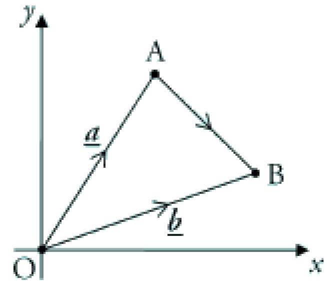
(b) $\frac{1}{2}\underline{\boldsymbol{m}} + \underline{\boldsymbol{n}} = \frac{1}{2}\begin{pmatrix} 2 \\ -4 \\ 6 \end{pmatrix} + \begin{pmatrix} -3 \\ 5 \\ -1 \end{pmatrix}$

$= \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} + \begin{pmatrix} -3 \\ 5 \\ -1 \end{pmatrix}$

$= \begin{pmatrix} -2 \\ 3 \\ 2 \end{pmatrix}$

Now attempt Exercise 1 from the Vectors booklet

Position Vectors & Magnitude:



- A **POSITION VECTOR** is a Vector joining a point,
  - $A(x\ ,y\ ,z)$, to the Origin, i.e. $\overrightarrow{OA} = \underline{a} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

VERY important

- The Vector joining any 2 points, A and B, can be found using the position Vectors: $\overrightarrow{AB} = \underline{b} - \underline{a}$
- The **MAGNITUDE** is written as $|\underline{\boldsymbol{u}}|$ or $|\overrightarrow{AB}|$ and can be calculated using the components:

$$|\underline{\boldsymbol{u}}| = \sqrt{x^2 + y^2 + z^2} \quad or \quad |\overrightarrow{AB}| = \sqrt{x^2 + y^2 + z^2}$$

For 2D simply miss out the z part!!

Examples:

5. P, Q and R are the points (3 , –2 , −5), (−3 , –4 , 5) and (–8 , 6 , 4) respectively.

a) State the components of the 3 position vectors $\underline{\boldsymbol{p}}$, $\underline{\boldsymbol{q}}$ & $\underline{\boldsymbol{r}}$.

$$\underline{p} = \begin{pmatrix} 3 \\ -2 \\ -5 \end{pmatrix} \qquad \underline{q} = \begin{pmatrix} -4 \\ -3 \\ 5 \end{pmatrix} \qquad \underline{r} = \begin{pmatrix} -8 \\ 6 \\ 4 \end{pmatrix}$$

b) State the components of the vectors $\overrightarrow{PR}$ & $\overrightarrow{QP}$

$$\overrightarrow{PR} = \underline{r} - \underline{p}$$
$$\overrightarrow{PR} = \begin{pmatrix} -8 \\ 6 \\ 4 \end{pmatrix} - \begin{pmatrix} 3 \\ -2 \\ -5 \end{pmatrix}$$
$$\overrightarrow{PR} = \begin{pmatrix} -11 \\ 8 \\ 9 \end{pmatrix}$$

$$\overrightarrow{QP} = \underline{p} - \underline{q}$$
$$\overrightarrow{QP} = \begin{pmatrix} 3 \\ -2 \\ -5 \end{pmatrix} - \begin{pmatrix} -4 \\ -3 \\ 5 \end{pmatrix}$$
$$\overrightarrow{QP} = \begin{pmatrix} 7 \\ 1 \\ -10 \end{pmatrix}$$

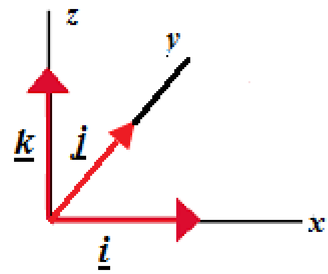
c) Calculate $|\overrightarrow{PR}|$ & $|\underline{q}|$

$$|\overrightarrow{PR}| = \sqrt{x^2 + y^2 + z^2}$$
$$|\overrightarrow{PR}| = \sqrt{(-11)^2 + 8^2 + 9^2}$$
$$|\overrightarrow{PR}| = \sqrt{121 + 64 + 81}$$
$$|\overrightarrow{PR}| = \sqrt{266}$$

$$|\underline{q}| = \sqrt{x^2 + y^2 + z^2}$$
$$|\underline{q}| = \sqrt{(-4)^2 + (-3)^2 + 5^2}$$
$$|\underline{q}| = \sqrt{16 + 9 + 25}$$
$$|\underline{q}| = \sqrt{50} \;=\; 5\sqrt{2}$$

Now attempt Exercise 2 & 3 from the Vectors booklet

Unit Vectors:



- A **UNIT VECTOR** is a vector with magnitude ONE.
- Any Vector can be expressed in terms of the Unit Vectors $\underline{\boldsymbol{i}} = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $\underline{\boldsymbol{j}} = \begin{pmatrix}0\\1\\0\end{pmatrix}$ and $\underline{\boldsymbol{l}} = \begin{pmatrix}0\\0\\1\end{pmatrix}$
- A **UNIT VECTOR** can be obtained from any vector by dividing each component by its magnitude:

i.e. $\underline{\boldsymbol{a}} = \begin{pmatrix}x\\y\\z\end{pmatrix}$, so the unit vector of $\underline{\boldsymbol{a}} = \frac{1}{|\underline{a}|}\begin{pmatrix}x\\y\\z\end{pmatrix}$

Examples:

6. **a)** Express the vector, $\underline{\boldsymbol{a}} = \begin{pmatrix}2\\-4\\-8\end{pmatrix}$ in terms of $\underline{\boldsymbol{i}}$, $\underline{\boldsymbol{j}}$, $\underline{\boldsymbol{k}}$: $2\underline{\boldsymbol{i}} - 4\underline{\boldsymbol{j}} - 8\underline{\boldsymbol{k}}$

b) Write the components of the vector $\underline{\boldsymbol{a}} = 5\underline{\boldsymbol{i}} - 3\underline{\boldsymbol{j}} + 2\underline{\boldsymbol{k}}$: $\underline{\boldsymbol{a}} = \begin{pmatrix}5\\-3\\2\end{pmatrix}$

c) Write the components of the vector $\underline{\boldsymbol{b}} = -7\underline{\boldsymbol{i}} + 11\underline{\boldsymbol{k}}$: $\underline{\boldsymbol{b}} = \begin{pmatrix}-7\\0\\11\end{pmatrix}$

d) Calculate the magnitude of the vector $\underline{\boldsymbol{c}} = 5\underline{\boldsymbol{i}} + 2\underline{\boldsymbol{j}} - \underline{\boldsymbol{k}}$

$$|\underline{c}| = \sqrt{x^2 + y^2 + z^2}$$
$$|\underline{c}| = \sqrt{5^2 + 2^2 + (-1)^2}$$
$$|\underline{c}| = \sqrt{25 + 4 + 1}$$
$$|\underline{c}| = \sqrt{30}$$

7. Find the components of the unit vector from the vector $\underline{\boldsymbol{a}} = \begin{pmatrix}12\\0\\-5\end{pmatrix}$

$$|\underline{a}| = \sqrt{x^2 + y^2 + z^2}$$
$$|\underline{a}| = \sqrt{12^2 + 0^2 + (-5)^2}$$
$$|\underline{a}| = \sqrt{144 + 0 + 25}$$
$$|\underline{a}| = \sqrt{169}$$
$$|\underline{a}| = 13$$

So the unit vector is $\frac{1}{13}\begin{pmatrix}12\\0\\-5\end{pmatrix}$

Now attempt Exercise 4 from the Vectors booklet

Collinearity:

- As seen earlier in the STRAIGHT LINE topic, 3 points are said to be COLLINEAR , i.e. lie on a straight line if the gradient between pairs of points are equal (PARALLEL) and if one of the points is common to both Gradients, the same is true for 3D Vectors.
- A vector is **PARALLEL** to another vector if it is a multiple of the first one, i.e. $\underline{v} = k\underline{u}$ Direction is not important!
- We cannot find Gradients in 3D, but $\underline{\boldsymbol{v}} = k\underline{\boldsymbol{u}}$ is the equivalent.

Examples:

8. Prove that the vectors, $\underline{\boldsymbol{a}} = \begin{pmatrix} 2 \\ -4 \\ -8 \end{pmatrix}$ and $\underline{\boldsymbol{b}} = \begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix}$ are parallel.

$\underline{\boldsymbol{a}} = \begin{pmatrix} 2 \\ -4 \\ -8 \end{pmatrix} = -2\begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix} = -2\underline{b}$ hence $\underline{\boldsymbol{a}}$ is parallel to $\underline{\boldsymbol{b}}$

9. **a)** Prove that the points $A(0 , -2 , 5)$, $B(4 , 2 , 3)$ and $C(10 , 8 , 0)$ are collinear.

$\overrightarrow{AB} = \underline{b} - \underline{a}$

$\overrightarrow{AB} = \begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 0 \\ -2 \\ 5 \end{pmatrix}$

$\overrightarrow{AB} = \begin{pmatrix} 4 \\ 4 \\ -2 \end{pmatrix}$

$\overrightarrow{AB} = 2\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$

$\overrightarrow{BC} = \underline{c} - \underline{b}$

$\overrightarrow{BC} = \begin{pmatrix} 10 \\ 8 \\ 0 \end{pmatrix} - \begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix}$

$\overrightarrow{BC} = \begin{pmatrix} 6 \\ 6 \\ -3 \end{pmatrix}$

$\overrightarrow{BC} = 3\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$

so $3\overrightarrow{AB} = 2\overrightarrow{BC}$

so $\overrightarrow{AB} = \frac{2}{3}\overrightarrow{BC}$, Therefore $\overrightarrow{AB}$ is parallel to $\overrightarrow{BC}$
and since B is a common point, A, B & C are collinear.

Statement MUST be written

b) Find the ratio $\overrightarrow{AB} : \overrightarrow{BC}$

$\overrightarrow{AB} = \frac{2}{3}\overrightarrow{BC} \implies \frac{\overrightarrow{AB}}{\overrightarrow{BC}} = \frac{2}{3}$, so $\overrightarrow{AB} : \overrightarrow{BC}$ is $2 : 3$

10. The points $P(3, 4, 7)$, $Q(m, 6, 3)$ and $R(-6, 1, 13)$ are collinear.

State the ratio in which Q divides PR and determine the value of m

$$\overrightarrow{PQ} = \underline{q} - \underline{p} \qquad\qquad \overrightarrow{QR} = \underline{r} - \underline{q}$$

$$\overrightarrow{PQ} = \begin{pmatrix} m \\ 6 \\ 3 \end{pmatrix} - \begin{pmatrix} 3 \\ 4 \\ 7 \end{pmatrix} \qquad\qquad \overrightarrow{QR} = \begin{pmatrix} -6 \\ 1 \\ 13 \end{pmatrix} - \begin{pmatrix} m \\ 6 \\ 3 \end{pmatrix}$$

$$\overrightarrow{PQ} = \begin{pmatrix} m-3 \\ 2 \\ -4 \end{pmatrix} \qquad\qquad \overrightarrow{QR} = \begin{pmatrix} -6-m \\ -5 \\ 10 \end{pmatrix}$$

Look at rows that only have numbers to find ratio

so $\quad -5\overrightarrow{PQ} = 2\overrightarrow{QR}$

Use the rows with variable

so $\quad \overrightarrow{PQ} = -\frac{2}{5}\overrightarrow{QR} \quad \Longrightarrow \quad \frac{\overrightarrow{PQ}}{\overrightarrow{QR}} = -\frac{2}{5}$, so $\overrightarrow{PQ} : \overrightarrow{QR}$ is $-2 : 5$

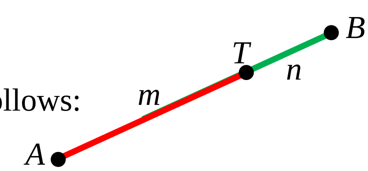
so $\quad -5(m-3) = 2(-6-m)$

$$-5m + 15 = -12 - 2m$$
$$-5m + 2m = -12 - 15$$
$$-3m = -27$$
$$m = 9$$

Now attempt Exercise 5 from the Vectors booklet

<u>Dividing Lines:</u>

- A point can split a line joining 2 other points in a given ratio as follows:

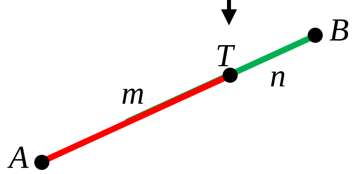


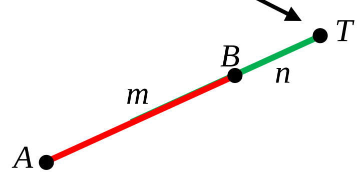
- So T splits the line AB in the ratio $m : n$

- If we know the coordinates of 2 points we can find the third point in 2 ways:
  - Algebraically using the fact that $\overrightarrow{AB} = \underline{b} - \underline{a}$
  - Or Using the SECTION FORMULA: $\underline{t} = \frac{n}{m+n}\underline{\boldsymbol{a}} + \frac{m}{m+n}\underline{\boldsymbol{b}}$

Choose which you prefer

- The point T can either split the line AB **<u>INTERNALLY</u>** or **<u>EXTERNALLY</u>** as shown below:





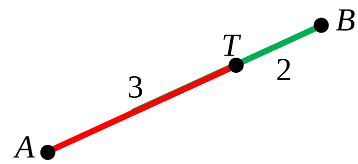
If no diagram is given then sketch one!!

Examples:

11. The point T divides the line AB in the ratio 3 : 2.

For the coordinates $A(-4 , 5 , 1)$ & $B(-24 , -10 , 26)$

find the coordinates of the point T.



Read the Question!!

Section Formula

$$\underline{t} = \frac{n}{m+n}\underline{\boldsymbol{a}} + \frac{m}{m+n}\underline{\boldsymbol{b}}$$

$$\underline{t} = \frac{2}{3+2}\begin{pmatrix}-4\\5\\1\end{pmatrix} + \frac{3}{3+2}\begin{pmatrix}-24\\-10\\26\end{pmatrix}$$

$$\underline{t} = \frac{2}{5}\begin{pmatrix}-4\\5\\1\end{pmatrix} + \frac{3}{5}\begin{pmatrix}-24\\-10\\26\end{pmatrix}$$

$$\underline{t} = \begin{pmatrix}-{}^{8}/_{5}\\2\\{}^{2}/_{5}\end{pmatrix} + \begin{pmatrix}-{}^{72}/_{5}\\-6\\{}^{78}/_{5}\end{pmatrix}$$

$$\underline{t} = \begin{pmatrix}-{}^{80}/_{5}\\-4\\{}^{80}/_{5}\end{pmatrix} = \begin{pmatrix}-16\\-4\\16\end{pmatrix}$$

Components

So T (–16 , –4 , 16)

Coordinates

Algebraically

$$2\overrightarrow{AT} = 3\overrightarrow{TB} \quad \Rightarrow \quad 2(\underline{t} - \underline{a}) = 3(\underline{b} - \underline{t})$$

$$\Rightarrow \quad 2\underline{t} - 2\underline{a} = 3\underline{b} - 3\underline{t}$$

$$\Rightarrow \quad 5\underline{t} = 3\underline{b} + 2\underline{a}$$

$$5\underline{t} = 3\begin{pmatrix}-24\\-10\\26\end{pmatrix} + 2\begin{pmatrix}-4\\5\\1\end{pmatrix}$$

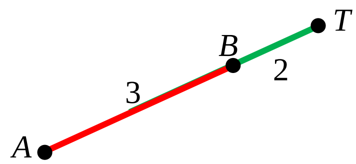
$$5\underline{t} = \begin{pmatrix}-72\\-30\\78\end{pmatrix} + \begin{pmatrix}-2\\10\\2\end{pmatrix}$$

$$5\underline{t} = \begin{pmatrix}-80\\-20\\80\end{pmatrix}$$

$$\underline{t} = \begin{pmatrix}-16\\-4\\16\end{pmatrix}$$

So T (–16 , –4 , 16)

12. The point B divides the line AT in the ratio 3 : 2.
For the coordinates $A(3, -1, 4)$ & $B(6, -4, 1)$
find the coordinates of the point T.



Algebraically

$$2\overrightarrow{AB} = 3\overrightarrow{BT} \implies 2(\underline{b} - \underline{a}) = 3(\underline{t} - \underline{b})$$
$$\implies 2\underline{b} - 2\underline{a} = 3\underline{t} - 3\underline{b}$$
$$\implies 3\underline{t} = 5\underline{b} - 2\underline{a}$$
$$3\underline{t} = 5\begin{pmatrix}6\\-4\\1\end{pmatrix} - 2\begin{pmatrix}3\\-1\\4\end{pmatrix}$$
$$3\underline{t} = \begin{pmatrix}30\\-20\\5\end{pmatrix} - \begin{pmatrix}6\\-2\\8\end{pmatrix}$$
$$3\underline{t} = \begin{pmatrix}24\\-18\\-3\end{pmatrix}$$
$$\underline{t} = \begin{pmatrix}8\\-6\\-1\end{pmatrix}$$ ← **Components**

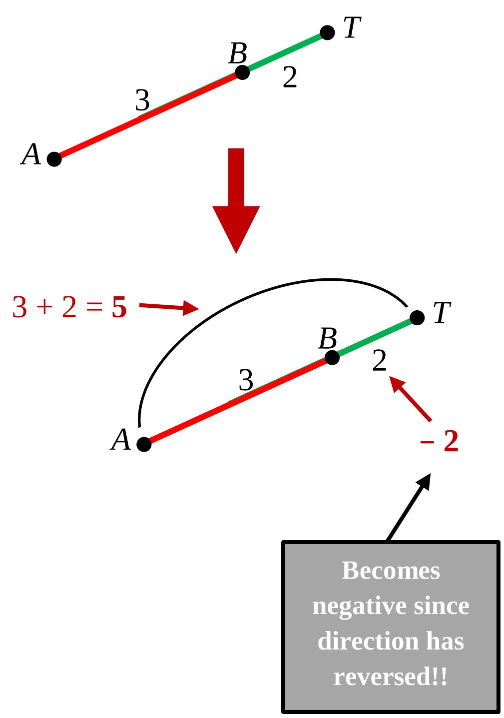
So T (8 , –6 , –1) ← **Coordinates**

When the point you are finding is external the Section Formula requires more work prior to the actual calculation!!

Section Formula

$$\underline{t} = \frac{n}{m+n}\underline{\boldsymbol{a}} + \frac{m}{m+n}\underline{\boldsymbol{b}}$$
$$\underline{t} = \frac{-2}{5+(-2)}\begin{pmatrix}3\\-1\\4\end{pmatrix} + \frac{5}{5+(-2)}\begin{pmatrix}6\\-4\\1\end{pmatrix}$$
$$\underline{t} = \frac{-2}{3}\begin{pmatrix}3\\-1\\4\end{pmatrix} + \frac{5}{3}\begin{pmatrix}6\\-4\\1\end{pmatrix}$$
$$\underline{t} = \begin{pmatrix}-2\\2/3\\-8/3\end{pmatrix} + \begin{pmatrix}10\\-20/3\\5/3\end{pmatrix}$$
$$\underline{t} = \begin{pmatrix}8\\-18/3\\-3/3\end{pmatrix} = \begin{pmatrix}8\\-6\\-1\end{pmatrix}$$ ← **Components**

So T (8 , –6 , –1) ← **Coordinates**

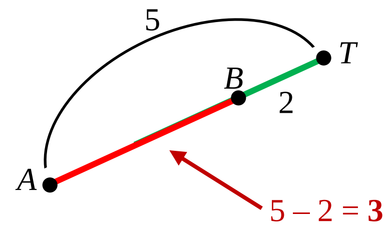


This question may be worded differently as follows but would be done exactly the same way as above.

12. The point T divides the line AB **externally** in the ratio 5 : 2.

For the coordinates $A(3\ ,-1\ ,4)$ & $B(6\ ,-4\ ,1)$

find the coordinates of the point T.



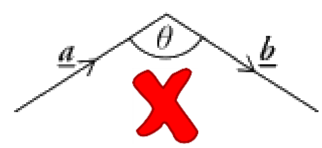
Now attempt Exercise 6 from the Vectors booklet

Scalar Product:

- The **SCALAR PRODUCT** is as close to multiplying vectors together as we can get!
- It is sometimes known as the **DOT PRODUCT** as it is written in the form: $\underline{\boldsymbol{a}}\,.\,\underline{\boldsymbol{b}}$
- The scalar product is a measure of how closely 2 vectors align, in terms of the directions they point.
- The Scalar (or Dot) Product can be calculated in 2 ways, using the:
  - Component form: $\underline{a}.\underline{b} = a_1 \times b_1 + a_2 \times b_2 + a_3 \times b_3$
  - Modulus form: $\underline{a}.\underline{b} = |\underline{a}||\underline{b}|\cos\theta$, where θ is the angle between the vectors $\underline{a}$ and $\underline{b}$
- To use the Modulus form both vectors must either be moving away from or towards the angle:

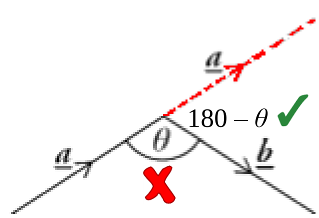
Both moving away from angle

Both moving towards the angle



$\underline{\boldsymbol{a}}$ is moving towards and $\underline{\boldsymbol{b}}$ is moving away from the angle.

- We can rearrange the 3rd diagram to allow us to find the Dot Product:



<u>Examples:</u>

13. Find $\underline{a}.\underline{b}$ given that $\underline{\boldsymbol{a}} = \begin{pmatrix} 5 \\ -3 \\ 2 \end{pmatrix}$ and $\underline{\boldsymbol{b}} = \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix}$

$$\underline{a}.\underline{b} = a_1 \times b_1 + a_2 \times b_2 + a_3 \times b_3$$
$$\underline{a}.\underline{b} = 5 \times 3 + (-3) \times 4 + 2 \times (-1)$$
$$\underline{a}.\underline{b} = 15 + (-12) + (-2) = 1$$

14. Find $\overrightarrow{PQ}.\overrightarrow{PR}$ given that $P(2\ ,-4\ ,9)$, $Q(3\ ,-1\ ,4)$ & $R(6\ ,-4\ ,1)$

$$\overrightarrow{PQ} = \underline{q} - \underline{p} \qquad \overrightarrow{PR} = \underline{r} - \underline{p}$$

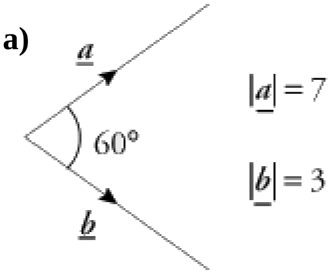
$$\overrightarrow{PQ} = \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix} - \begin{pmatrix} 2 \\ -4 \\ 9 \end{pmatrix} \qquad \overrightarrow{PR} = \begin{pmatrix} 6 \\ -4 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ -4 \\ 9 \end{pmatrix}$$

$$\overrightarrow{PQ} = \begin{pmatrix} 1 \\ 3 \\ -5 \end{pmatrix} \qquad \overrightarrow{PR} = \begin{pmatrix} 4 \\ 0 \\ -8 \end{pmatrix}$$

$$\overrightarrow{PQ}.\overrightarrow{PR} = a_1 \times b_1 + a_2 \times b_2 + a_3 \times b_3$$
$$\overrightarrow{PQ}.\overrightarrow{PR} = 1 \times 4 + 3 \times 0 + (-5) \times (-8)$$
$$\overrightarrow{PQ}.\overrightarrow{PR} = 4 + 0 + 40 = 44$$

15. For each diagram below calculate $\underline{a}.\underline{b}$:

a)



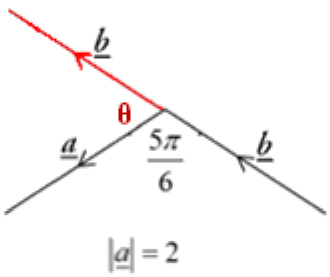
$\underline{a}.\underline{b} = |\underline{a}||\underline{b}|\cos\theta$

$\underline{a}.\underline{b} = 7 \times 3 \times \cos 60$

$\underline{a}.\underline{b} = 21 \times 0.5$

$\underline{a}.\underline{b} = 10.5$

b)



$$\underline{b} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$$

$\frac{5\pi}{6} = 150°$

$\theta = 180 - 150$

$\theta = 30$

$|\underline{\boldsymbol{b}}| = \sqrt{x^2 + y^2 + z^2}$

$|\underline{\boldsymbol{b}}| = \sqrt{3^2 + (-1)^2 + 2^2}$

$|\underline{\boldsymbol{b}}| = \sqrt{9 + 1 + 4}$

$|\underline{\boldsymbol{b}}| = \sqrt{14}$

$\underline{a}.\underline{b} = |\underline{a}||\underline{b}|\cos\theta$

$\underline{a}.\underline{b} = 2 \times \sqrt{14} \times \cos 30$

$\underline{a}.\underline{b} = 2\sqrt{14} \times \frac{\sqrt{3}}{2}$

$\underline{a}.\underline{b} = \sqrt{14} \times \sqrt{3} = \sqrt{42}$

Now attempt Exercise 7 from the Vectors booklet

Angle Between 2 Vectors:

- We can rearrange the Scalar Product to allow us to be able to find the Angle between the 2 vectors:

$$\cos\theta = \frac{\underline{a}.\underline{b}}{|\underline{a}||\underline{b}|} = \frac{a_1 \times b_1 + a_2 \times b_2 + a_3 \times b_3}{|\underline{a}||\underline{b}|}$$

Examples:

16. Calculate the angle, θ, between the 2 vectors, $\underline{\boldsymbol{m}} = 3\underline{\boldsymbol{i}} + 4\underline{\boldsymbol{j}} - 2\underline{\boldsymbol{k}}$ and $\underline{\boldsymbol{n}} = 4\underline{\boldsymbol{i}} + \underline{\boldsymbol{j}} + 3\underline{\boldsymbol{k}}$

$$\underline{\boldsymbol{m}} = 3\underline{\boldsymbol{i}} + 4\underline{\boldsymbol{j}} - 2\underline{\boldsymbol{k}}$$

$$\underline{\boldsymbol{m}} = \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix}$$

$$\underline{\boldsymbol{n}} = \mathbf{4}\underline{\boldsymbol{i}} + \underline{\boldsymbol{j}} + \mathbf{3}\underline{\boldsymbol{k}}$$

$$\underline{\boldsymbol{n}} = \begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix}$$

$$|\underline{\boldsymbol{m}}| = \sqrt{x^2 + y^2 + z^2}$$

$$|\underline{\boldsymbol{m}}| = \sqrt{3^2 + 4^2 + (-2)^2}$$

$$|\underline{\boldsymbol{m}}| = \sqrt{9 + 16 + 4}$$

$$|\underline{\boldsymbol{m}}| = \sqrt{29}$$

$$|\underline{\boldsymbol{n}}| = \sqrt{x^2 + y^2 + z^2}$$

$$|\underline{\boldsymbol{n}}| = \sqrt{4^2 + 1^2 + 3^2}$$

$$|\underline{\boldsymbol{n}}| = \sqrt{16 + 1 + 9}$$

$$|\underline{\boldsymbol{n}}| = \sqrt{26}$$

$$\underline{m}.\underline{n} = a_1 \times b_1 + a_2 \times b_2 + a_3 \times b_3$$

$$\underline{m}.\underline{n} = 3 \times 4 + 4 \times 1 + (-2) \times 3$$

$$\underline{m}.\underline{n} = 12 + 4 + (-6) = 10$$

$$cos\,\theta = \frac{\underline{m}.\underline{n}}{|\underline{m}||\underline{n}|}$$

$$cos\,\theta = \frac{10}{\sqrt{29} \times \sqrt{26}}$$

$$\theta = \cos^{-1} 0.364\ldots$$

$$\theta = 68.64°$$

17. Find $\angle DEF$ when $D(2\ ,5\ ,1)$, $E(-3\ ,3\ ,4)$ & $F(1\ ,-7\ ,2)$

As both vectors MUST travel in the same direction start from the outsides of the angle's name and work in! (you can also start from the inside and work out)!!

$$\overrightarrow{DE} = \underline{e} - \underline{d}$$

$$\overrightarrow{DE} = \begin{pmatrix} -3 \\ 3 \\ 4 \end{pmatrix} - \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix}$$

$$\overrightarrow{DE} = \begin{pmatrix} -5 \\ -2 \\ 3 \end{pmatrix} = \underline{a}$$

$$\overrightarrow{FE} = \underline{e} - \underline{f}$$

$$\overrightarrow{FE} = \begin{pmatrix} -3 \\ 3 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ -7 \\ 2 \end{pmatrix}$$

$$\overrightarrow{FE} = \begin{pmatrix} -4 \\ 10 \\ 2 \end{pmatrix} = \underline{b}$$

$$|\underline{\boldsymbol{a}}| = \sqrt{x^2 + y^2 + z^2}$$

$$|\underline{\boldsymbol{a}}| = \sqrt{(-5)^2 + (-2)^2 + 3^2}$$

$$|\underline{\boldsymbol{a}}| = \sqrt{25 + 4 + 9}$$

$$|\underline{\boldsymbol{a}}| = \sqrt{38}$$

$$|\underline{\boldsymbol{b}}| = \sqrt{x^2 + y^2 + z^2}$$

$$|\underline{\boldsymbol{b}}| = \sqrt{(-4)^2 + 10^2 + 2^2}$$

$$|\underline{\boldsymbol{b}}| = \sqrt{16 + 100 + 4}$$

$$|\underline{\boldsymbol{b}}| = \sqrt{120}$$

$$\underline{a}.\underline{b} = a_1 \times b_1 + a_2 \times b_2 + a_3 \times b_3$$

$$\underline{a}.\underline{b} = (-5) \times (-4) + (-2) \times 10 + 3 \times 2$$

$$\underline{a}.\underline{b} = 20 + (-20) + 6 = 6$$

$$cos\,\theta = \frac{\underline{a}.\underline{b}}{|\underline{a}||\underline{b}|}$$

$$cos\,\theta = \frac{6}{\sqrt{39} \times \sqrt{120}}$$

$$\theta = \cos^{-1} 0.0877\ldots$$

$$\theta = 84.97°$$

Now attempt Exercise 8 from the Vectors booklet

Perpendicular Vectors:

- If vectors $\underline{\boldsymbol{a}}$ and $\underline{\boldsymbol{b}}$ are perpendicular then $\underline{a}.\underline{b} = 0$

Examples:

18. Show that vectors $\underline{\boldsymbol{a}}$ and $\underline{\boldsymbol{b}}$ are perpendicular when $\underline{\boldsymbol{a}} = 3\underline{\boldsymbol{i}} - 5\underline{\boldsymbol{j}} + \underline{\boldsymbol{k}}$ and $\underline{\boldsymbol{b}} = 4\underline{\boldsymbol{i}} + 3\underline{\boldsymbol{j}} + 3\underline{\boldsymbol{k}}$

$\underline{\boldsymbol{a}} = \boldsymbol{3\underline{i} - 5\underline{j} + \underline{k}}$

$\underline{\boldsymbol{b}} = \boldsymbol{4\underline{i} + 3\underline{j} + 3\underline{k}}$

$$\underline{\boldsymbol{a}} = \begin{pmatrix} 3 \\ -5 \\ 1 \end{pmatrix} \qquad \underline{\boldsymbol{b}} = \begin{pmatrix} 4 \\ 3 \\ 3 \end{pmatrix}$$

$\underline{a}.\underline{b} = a_1 \times b_1 + a_2 \times b_2 + a_3 \times b_3$

$\underline{a}.\underline{b} = 3 \times 4 + (-5) \times 3 + 1 \times 3$

$\underline{a}.\underline{b} = 12 + (-15) + 3 = 0$ Since $\underline{a}.\underline{b} = 0$ $\underline{\boldsymbol{a}}$ is perpendicular to $\underline{\boldsymbol{b}}$.

19. Find the value of k if the direct line segments $\overrightarrow{DE} = \begin{pmatrix} 3 \\ -2 \\ k \end{pmatrix}$ and $\overrightarrow{FG} = \begin{pmatrix} -4 \\ -5 \\ 2 \end{pmatrix}$ are perpendicular.

Since $\overrightarrow{DE}$ is perpendicular to $\overrightarrow{FG}$

$$\begin{aligned} \overrightarrow{DE}.\overrightarrow{FG} &= 0 \\ a_1 \times b_1 + a_2 \times b_2 + a_3 \times b_3 &= 0 \\ 3 \times (-4) + (-2) \times (-5) + k \times 2 &= 0 \\ -12 + 15 + 2k &= 0 \\ 3 + 2k &= 0 \\ 2k &= -3 \\ k &= -1.5 \end{aligned}$$

Now attempt Exercise 9 from the Vectors booklet

Scalar Product Properties:

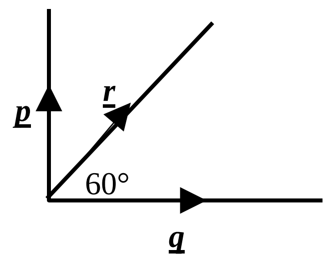
- For Vectors vectors ***a*** , ***b*** and ***c*** the following 3 properties exist:
  - $\underline{a}.\underline{b} = \underline{b}.\underline{a}$
  - $\underline{a}.(\underline{b} + \underline{c}) = \underline{a}.\underline{b} + \underline{a}.\underline{c}$
  - $\underline{a}.\underline{a} = |\underline{a}|^2$

Examples:

20. Given that vectors ***p*** and ***q*** are perpendicular

and $|\underline{p}| = 2$, $|\underline{q}| = 3$, $|\underline{r}| = \sqrt{3}$

calculate the exact value of ***p*** . (***q*** + ***r***)



Angle between vectors ***p*** and ***r*** is: 90 – 60 = 30°

$\underline{p} . (\underline{q} + \underline{r}) = \underline{p} . \underline{q} + \underline{p} . \underline{r}$

$= 0 + |\underline{p}| \, |\underline{r}| \cos\theta$

Since vectors ***p*** and ***q*** are perpendicular. ***p*** . ***q*** = 0

$= 2 \times \sqrt{3} \times \cos 30$

$= 2\sqrt{3} \times \frac{\sqrt{3}}{2}$

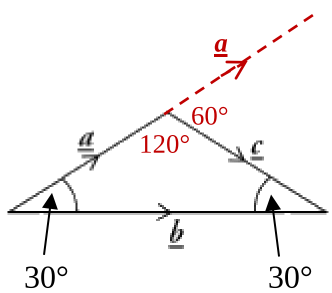
$= \frac{2 \times \sqrt{3} \times \sqrt{3}}{2}$

$= \sqrt{9}$

$= 3$

21. For the diagram opposite and $|\underline{a}| = 4$, $|\underline{b}| = 4\sqrt{3}$

calculate the exact value of ***a*** . (***a*** + ***b*** + ***c***)



$\underline{a} . (\underline{a} + \underline{b} + \underline{c}) = \underline{a} . \underline{a} + \underline{a} . \underline{b} + \underline{a} . \underline{c}$

$= 4^2 + 4 \times 4\sqrt{3} \times \cos 30 + 4 \times 4 \times \cos 60$

$= 16 + 16\sqrt{3} \times \frac{\sqrt{3}}{2} + 16 \times \frac{1}{2}$

$= 16 + \frac{16\sqrt{9}}{2} + 8$

$= 16 + 24 + 8$

$= 48$

Now attempt Exercise 10 from the Vectors booklet

Adapting Known Formulae:

- The following formulae can be used in 3 Dimensions by adding a *z* part to them:
  - Distance Formula: $\sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}$
  - Midpoint Formula: $\left(\frac{x_A+x_B}{2} \;,\; \frac{y_A+y_B}{2} \;,\; \frac{z_A+z_B}{2}\right)$

Not used much in vectors!!

- The Position Vector of the midpoint can be found using the formula: $M = \frac{1}{2}(\underline{\boldsymbol{a}} + \underline{\boldsymbol{b}})$

Examples:

22. Find the coordinates of the midpoint of $\overrightarrow{ST}$ where S (8 , 3 , −6) & T (−2 , 7 , 0)

Midpoint Formula

$$M = \left(\frac{x_A+x_B}{2} \;,\; \frac{y_A+y_B}{2} \;,\; \frac{z_A+z_B}{2}\right)$$

$$M = \left(\frac{8+(-2)}{2} \;,\; \frac{3+7}{2} \;,\; \frac{-6+0}{2}\right)$$

$$M = \left(\frac{6}{2} \;,\; \frac{10}{2} \;,\; \frac{-6}{2}\right)$$

$$M = (3 \;,\; 5 \;, -3)$$

OR

Position Vector Midpoint

$$M = \frac{1}{2}\left[\begin{pmatrix} 8 \\ 3 \\ -6 \end{pmatrix} + \begin{pmatrix} -2 \\ 7 \\ 0 \end{pmatrix}\right]$$

$$M = \frac{1}{2}\begin{pmatrix} 6 \\ 10 \\ -6 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ -3 \end{pmatrix}$$

Vector Components

$$M = (3 \;,\; 5 \;, -3)$$

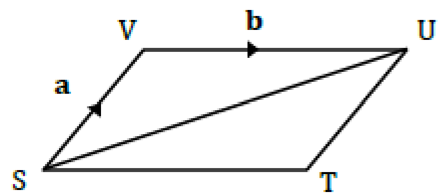
Coordinates

Vector Journeys:

- A Vector Journey is a description of the path from the beginning to the end of the vector.
- Vector Journeys can be described either by their components or by combining other vectors.

Examples:

23. a) Express the vector $\overrightarrow{SU}$ in terms of $\underline{\boldsymbol{a}}$ and $\underline{\boldsymbol{b}}$.



$$\overrightarrow{SU} = S \rightarrow V \rightarrow U$$

Fist and Last letter should always match!!

$$= \overrightarrow{SV} + \overrightarrow{VU}$$

$$= \underline{a} + \underline{b}$$

b) Express the vector $\overrightarrow{VT}$ in terms of $\underline{\boldsymbol{a}}$ and $\underline{\boldsymbol{b}}$.

$$\overrightarrow{VT} = V \rightarrow U \rightarrow T$$

$$= \overrightarrow{VU} + \overrightarrow{UT}$$

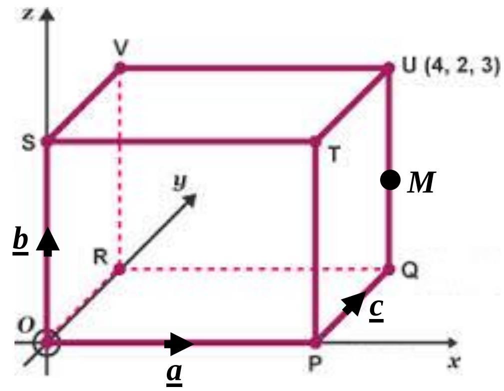
$$= \underline{b} + (-\underline{a})$$

$$= \underline{b} - \underline{a}$$

24. The shape opposite is a cuboid with a vertex at the origin and $U(4\,,2\,,3)$.

M is the midpoint of $\overrightarrow{QU}$

$\overrightarrow{OP} = \underline{\boldsymbol{a}}\,, \overrightarrow{OS} = \underline{\boldsymbol{b}}$ and $\overrightarrow{PQ} = \underline{\boldsymbol{c}}$



a) State the coordinates of the points

(i) P (4 , 0 , 0)

(ii) V (0 , 2 , 3)

(iii) M (4 , 2 , 1.5)

b) Express the vector $\overrightarrow{QS}$ in terms of $\underline{\boldsymbol{a}}\,, \underline{\boldsymbol{b}}$ and $\underline{\boldsymbol{c}}$

$$\overrightarrow{QS} = Q \rightarrow U \rightarrow V \rightarrow S$$

$$\overrightarrow{QS} = \overrightarrow{QU} + \overrightarrow{UV} + \overrightarrow{VS}$$

$$\overrightarrow{QS} = \underline{\boldsymbol{b}} + (-\underline{\boldsymbol{a}}) + (-\underline{\boldsymbol{c}})$$

$$\overrightarrow{QS} = \underline{\boldsymbol{b}} - \underline{\boldsymbol{a}} - \underline{\boldsymbol{c}}$$

c) Express the vector $\overrightarrow{SM}$ in terms of $\underline{\boldsymbol{a}}\,, \underline{\boldsymbol{b}}$ and $\underline{\boldsymbol{c}}$

$$\overrightarrow{SM} = S \rightarrow T \rightarrow U \rightarrow M$$

$$\overrightarrow{SM} = \overrightarrow{ST} + \overrightarrow{TU} + \overrightarrow{UM}$$

$$\overrightarrow{SM} = \underline{\boldsymbol{a}} + \underline{\boldsymbol{c}} + \tfrac{1}{2}(-\underline{\boldsymbol{b}})$$

$$\overrightarrow{SM} = \underline{\boldsymbol{a}} + \underline{\boldsymbol{c}} - \tfrac{1}{2}\underline{\boldsymbol{b}}$$

Now attempt Exercises 11 − 13 from the Vectors booklet