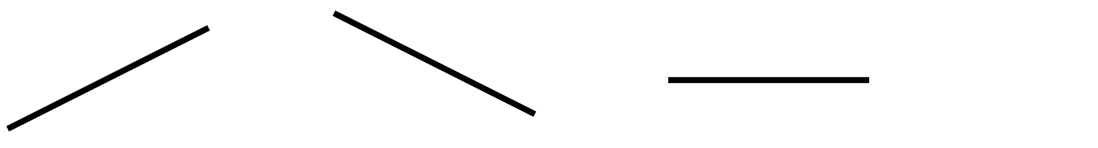
H Straight Line

SPTA Mathematics – Higher Notes

Gradient – From National 5:

- Gradient is a measure of a lines slope, the greater the gradient the more steep its slope and vice versa.
- We use the letter *m* to represent Gradient.
- Gradient of a Straight Line joining 2 points $A(x_A, y_A)$ and $B(x_B, y_B)$ is given by: $m_{AB} = \frac{y_B - y_A}{x_B - x_A}$
- Two lines with the **SAME** Gradient are said to be **PARALLEL**.
- Gradient, m, can be found from a line's equation by rearranging it into the form $y = mx + c$
- The Gradient of a line lets us know the direction of the line as follows:



Slopes UP from Left to Right	Slopes DOWN from Left to Right	HORIZONTAL	VERTICAL
Positive Gradient	Negative Gradient	Zero Gradient	Undefined Gradient
$y = mx + c$	$y = mx + c$	$y = b$	$x = a$

Examples:

1. Find the Gradient of the Straight Line joining the points $A(0, -4)$ and $B(4, 2)$

$x_A \quad y_A \qquad x_B \quad y_B$

$$m_{AB} = \frac{y_B - y_A}{x_B - x_A} = \frac{2-(-4)}{4-0} = \frac{6}{4} = \frac{3}{2}$$

It is a good idea to label your points.

2. The line through the points $P(-2, 5)$ and $Q(7, a)$ has gradient $^4/_3$. What is the value of a?

$x_P \quad y_P \qquad x_Q \quad y_Q$

$$m_{PQ} = \frac{y_Q - y_P}{x_Q - x_P} = \frac{a-5}{7-(-2)}$$

$$\Rightarrow \frac{a-5}{9} = \frac{4}{3}$$

$$a - 5 = \frac{36}{3}$$

$$a - 5 = 12 \Rightarrow a = 17$$

3. Find the Gradient of the line parallel to $2y + 3x - 5 = 0$

$2y + 3x - 5 = 0$

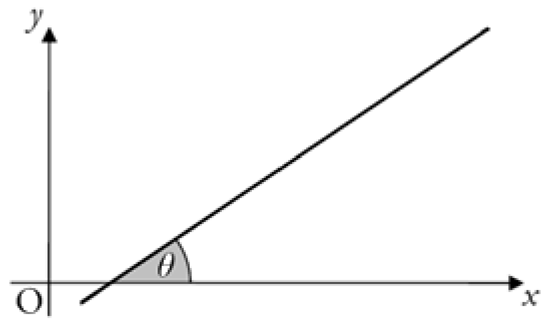
$2y = -3x + 5$

$y = -{}^{3}/_{2}\,x + {}^{5}/_{2}$

so $m = -{}^{3}/_{2}$ hence Parallel Gradient is $-{}^{3}/_{2}$

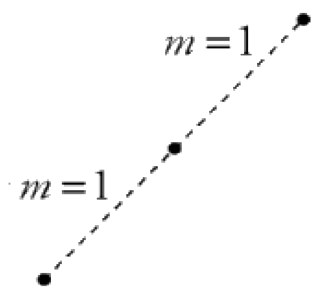
<u>Gradient – New Stuff:</u>

- The Gradient of a line can also be found using $m = \text{Tan}\,\theta$, where θ is the angle the line makes with the **<u>positive direction of the *x*-axis</u>** as follows:

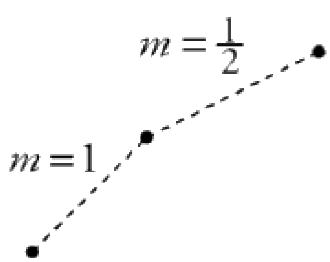


- If 2 lines, with Gradients m_1 and m_2, are perpendicular (meet at 90°) then: $m \times m_{perp} = -1$

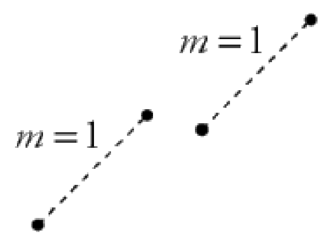
  So if you know the gradient of one line you can find the Gradient of the perpendicular line by inverting (flipping) it and changing the sign.
- For horizontal lines, i.e. $m = 0$ then the perpendicular line will be vertical with Undefined Gradient and vice versa.
- Points which lie on the same straight line are said to be collinear as follows:



Equal Gradients with
a common point
COLLINEAR



Unequal Gradients with
a common point
NOT Collinear



Equal Gradients but
NO common point
NOT Collinear

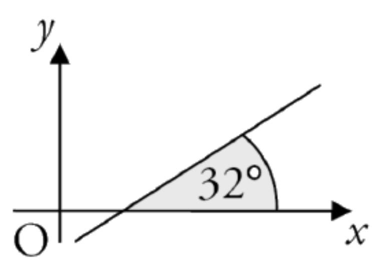
<u>Examples:</u>

4. Calculate the Gradient of the straight line opposite:

$m = \text{Tan}\,\theta$

$= \text{Tan}\,32°$

$= 0.62$ to 2 d.p.



5. Find the angle that the line through the points $P(\overset{x_P}{4}, \overset{y_P}{-3})$ and $Q(\overset{x_Q}{1}, \overset{y_Q}{6})$ makes with the positive direction of the x-axis.

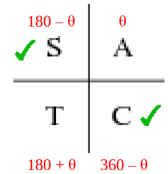
$$m_{PQ} = \frac{y_Q - y_P}{x_Q - x_P} = \frac{6-(-3)}{1-4} = \frac{9}{-3} = -3$$

$\Rightarrow m = \text{Tan}\ \theta$

$-3 = \text{Tan}\ \theta$

$\theta = \text{Tan}^{-1}(-3)$

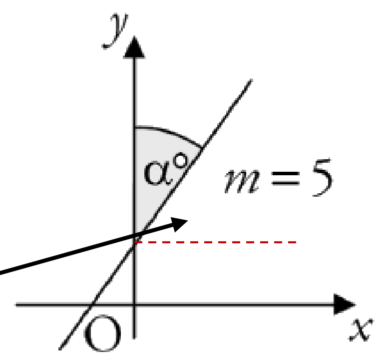
$\theta = 180 - 71.57° = 108.43°$ to 2 d.p.



Remember if the Gradient is a negative, do not enter this into your calculator. Use the positive gradient along with the CAST diagram from National 5 to find the correct angle.

The Angle will be less than 180°

6. Find the angle, α, marked in the diagram opposite:



$m = \text{Tan}\ \theta$

$5 = \text{Tan}\ \theta$

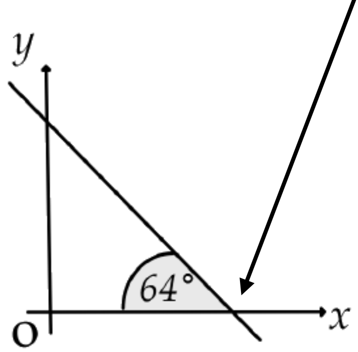
$\theta = \text{Tan}^{-1}\ 5$

$\theta = 78.69°$ to 2 d.p.

so $\alpha = 90 - 78.69$

$\alpha = 11.31°$ to 2 d.p.

m = Tan θ uses this angle.

7. 

Find the Gradient of the line shown.

$\theta = 180 - 64$

$\theta = 116°$

so $m = \text{Tan}\ \theta$

$m = \text{Tan}\ 116$

$m = -2.05$ to 2 d.p.

Now attempt Exercise 1 from the Straight Line booklet

8. State the perpendicular gradient of:

a) $m = -3$, since perpendicular $m \times m_{perp} = -1$

so $m_{perp} = {}^{1}/_{3}$

(b) $m = {}^{2}/_{5}$, since perpendicular $m \times m_{perp} = -1$

so $m_{perp} = -{}^{5}/_{2}$

Statement MUST be written.

9. Find the Gradient of the line perpendicular to the line joining the points $S(1, -2)$ and $T(-4, 5)$.

$x_S \quad y_S \qquad x_T \quad y_T$

$$m_{ST} = \frac{y_T - y_S}{x_T - x_S}$$

$$= \frac{5-(-2)}{-4-1}$$

$$= -\frac{7}{5}$$

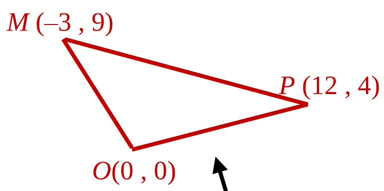
Since perpendicular

$m_{ST} \times m_{perp} = -1$

so $m_{perp} = {}^5/_7$

10. The points $M(-3, 9)$, $P(12, 4)$ and the Origin form a triangle, show that it's a right angled triangle.

$x_M \quad y_M \qquad x_P \quad y_P$



A quick sketch may make the question clearer.

$$m_{MP} = \frac{y_P - y_M}{x_P - x_M} = \frac{4-9}{12-(-3)} = \frac{-5}{15} = -\frac{1}{3}$$

$$m_{MO} = \frac{y_O - y_M}{x_O - x_M} = \frac{0-9}{0-(-3)} = \frac{-9}{3} = -3$$

$$m_{PO} = \frac{y_O - y_P}{x_O - x_P} = \frac{0-4}{0-12} = \frac{-5}{-12} = \frac{1}{3}$$

Since $m_{MO} \times m_{PO} = -1$, *MO* is perpendicular to *PO*
Therefore Triangle *MOP* is right angled at *O*.

11. Show that the points $P(-6, -1)$, $Q(0, 2)$ and $R(8, 6)$ are collinear.

$x_P \quad y_P \qquad x_Q \quad y_Q \qquad x_R \quad y_R$

$$m_{PQ} = \frac{y_Q - y_P}{x_Q - x_P} = \frac{2-(-1)}{0-(-6)} = \frac{3}{6} = \frac{1}{2}$$

$$m_{QR} = \frac{y_R - y_Q}{x_R - x_Q} = \frac{6-2}{8-0} = \frac{4}{8} = \frac{1}{2}$$

Statements MUST be written to gain full marks.

Since $m_{PQ} = m_{QR}$, *PQ* is parallel to *QR* and
Since *Q* is a common point, *P*, *Q*, *R* are collinear.

Now attempt Exercise 2 from the Straight Line booklet

Straight Line Formulae:

- As well as the 2 Gradient formulae mentioned above you will need to remember the following Formulae as they will **NOT** be given on your Formulae Sheet in the exam.
- From National 5 you should already know the Straight Line Formula: $y - b = m(x - a)$
- A line crosses the x–axis when $y = 0$ and crosses the y–axis when $x = 0$
- Remember to be able to find the equation of a Straight Line you need to know 2 things: A Point on the line and the Gradient of the line or 2 points on the line.
- The Straight Line Formulae will not work if the Gradient is undefined (Denominator = 0), i.e. a Vertical Line which has equation $x = a$
- The distance between 2 points $A\ (x_A , y_A)$ and $B\ (x_B , y_B)$ is given by the Distance Formula: $AB = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2}$
- The Midpoint between 2 points $A\ (x_A , y_A)$ and $B\ (x_B , y_B)$ is given by the Midpoint Formula: $\text{Mid}_{AB} = \left(\frac{x_A + x_B}{2} , \frac{y_A + y_B}{2}\right)$

Examples:

12. Calculate the length of the line joining the points $A(1 , -2)$ and $B(-3 , 6)$.

$x_A \quad y_A \qquad x_B \quad y_B$

$$AB = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2}$$
$$AB = \sqrt{(-3 - 1)^2 + (6 - (-2))^2}$$
$$AB = \sqrt{(-4)^2 + 8^2}$$
$$AB = \sqrt{16 + 64}$$
$$AB = \sqrt{80}$$
$$AB = \sqrt{16 \times 5} = 4\sqrt{5} \text{ units}$$

Preferable to leave as a SURD rather than a decimal

13. Find the midpoint of the 2 coordinates in Example 12 above.

$$\text{Mid}_{AB} = \left(\frac{x_A + x_B}{2} , \frac{y_A + y_B}{2}\right)$$
$$\text{Mid}_{AB} = \left(\frac{1 + (-3)}{2} , \frac{-2 + 6}{2}\right)$$
$$\text{Mid}_{AB} = \left(\frac{-2}{2} , \frac{4}{2}\right) = (-1 , 2)$$

Now attempt Exercises 3 & 4 from the Straight Line booklet

14. a) Find the equation of the straight line joining the points $A(-1, -3)$ and $B(2, 3)$

$x_A \quad y_A \qquad x_B \quad y_B$

$$m_{AB} = \frac{y_B - y_A}{x_B - x_A} = \frac{3-(-3)}{2-(-1)} = \frac{6}{3} = 2$$

$$y - b = m(x - a)$$

$$y - 3 = 2(x - 2)$$

$$y - 3 = 2x - 4$$

$$y = 2x - 1 \quad \text{or} \quad y - 2x = -1 \quad \text{or} \quad y - 2x + 1 = 0$$

You can sub in either point above to get the same equation.

The final equation can be left in any format.

b) Does the point $T(-4, -9)$ lie on the line AB above?

$x \quad y$

$y = 2x - 1$
$y = 2(-4) - 1$
$y = -8 - 1$
$y = -9$ hence the point T lies on the line $y = 2x - 1$

15. Find the equation of the straight line joining the points $A(2, -1)$ and $B(2, 5)$

$x_A \quad y_A \qquad x_B \quad y_B$

$$m_{AB} = \frac{y_B - y_A}{x_B - x_A} = \frac{5-(-1)}{2-2} = \frac{6}{0} = Undefined$$

Vertical line.

$$x = a$$

$$x = 2$$

16. a) Find the Equation of the line perpendicular to the line joining the points $S(1, -2)$ and $T(-4, 5)$ and passing through the point $U(-3, 2)$.

$x_S \quad y_S \qquad x_T \quad y_T$

$$m_{ST} = \frac{y_T - y_S}{x_T - x_S} = \frac{5-(-2)}{-4-1} = -\frac{7}{5}$$

Since perpendicular

$m_{ST} \times m_{perp} = -1$

so $m_{perp} = {}^5/_7$

$$y - b = m(x - a)$$

$$y - 2 = {}^5/_7(x - (-3))$$

$$7y - 14 = 5x + 15$$

$$7y = 5x + 29$$

b) Find the coordinates of where the line crosses the 2 axis.

Cuts the x–axis when $y = 0$

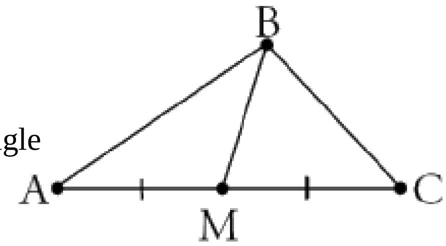
$7y = 5x + 29$
$7(0) = 5x + 29$
$0 = 5x + 29$
$29 = 5x$ $({}^{29}/_5, 0)$

Cuts the y–axis when $x = 0$

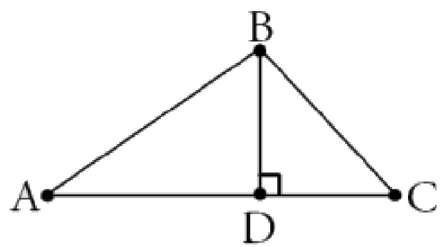
$7y = 5x + 29$
$7y = 5(0) + 29$
$7y = 29$
$7y = 29$ $(0, {}^{29}/_7)$

Now attempt Exercises 5A & 5B from the Straight Line booklet

3 Special Lines:

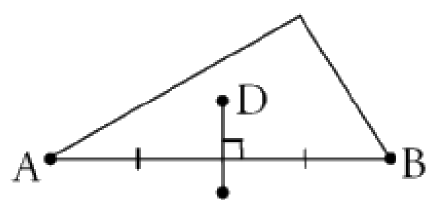


- A **MEDIAN** is a straight Line through a Vertex of a Triangle to the **Midpoint** of the opposite side as shown here:
- To find the equation of the MEDIAN, first find the Midpoint, then the Gradient of *BM* and then find the equation using either *Pt B* or *Pt M*.
- A Triangle has 3 Medians which all intersect at the same point (**CONCURRENT**). This point is called the **CENTROID**

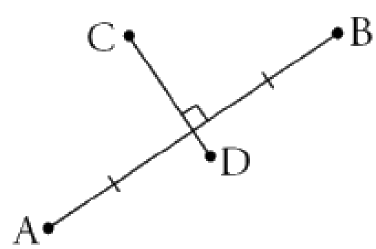


- An **ALTITUDE** is a straight Line through a Vertex of a Triangle **Perpendicular** to the opposite side as shown here:
- To find the equation of the Altitude, first find the the Gradient of *AC*, flip it and then find the equation using *Pt B*.

  There is NO need to find *Pt D* in order to find the Altitude!
- A Triangle has 3 Altitudes which also intersect at the same point (**CONCURRENT**). This point is called the **ORTHOCENTRE**



- A **PERPENDICULAR BISECTOR** is a straight Line which cuts through the Midpoint of another line at right angles as shown here:
- Perpendicular Bisectors do **NOT** need to involve Triangles.



- To find the equation of the Perpendicular Bisectors, first find the the Gradient of *AB*, flip it, now find the Midpoint of *AB* and then find the equation using the midpoint.
- When two lines cross over they are said to Intersect. We can find the Point of Intersection, POI, by using Simultaneous Equations which you saw in National 5.
- There are 3 ways of finding the POI, Elimination, Substitution and by Equating.

  If a line crosses the other line in the middle it is said to **BISECT** the line.
- Again Triangle has 3 Perpendicular Bisectors which are also **CONCURRENT**. This point is called the **CIRCUMCENTRE**. The Circumcentre is the centre of a circle passing through all 3 vertices of a triangle.

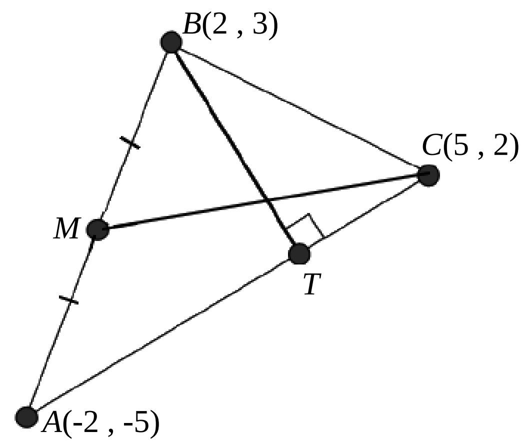
Examples:

17. a) Find the equation of the MEDIAN from *Pt C*.

b) Find the equation of the ALTITUDE from *Pt B*.

c) Find the PERPENDICULAR BISECTOR of *BC*.

d) Find the point where the MEDIAN & ALTITUDE above intersect.



a) $\text{Mid}_{AB} = \left(\frac{x_A+x_B}{2} \ , \ \frac{y_A+y_B}{2}\right)$ $\quad m_{MC} = \frac{y_C-y_M}{x_C-x_M}$ $\quad y - b = m(x - a)$

$\text{Mid}_{AB} = \left(\frac{-2+2}{2} \ , \ \frac{-5+3}{2}\right)$ $\quad = \frac{2-(-1)}{5-0}$ $\quad y - 2 = {}^3/_5(x - 5)$

$\text{Mid}_{AB} = \left(\frac{0}{2} \ , \ \frac{-2}{2}\right) = (0\,, -1)$ $\quad = \frac{3}{5}$ $\quad 5y - 10 = 3x - 15$

$5y - 3x = -5$

Now attempt Exercise 6 from the Straight Line booklet

b) $m_{AC} = \frac{y_C-y_A}{x_C-x_A}$ $\quad$ Since perpendicular $\quad y - b = m(x - a)$

$= \frac{2-(-5)}{5-(-2)}$ $\quad m_{AC} \text{ x } m_{perp} = -1$ $\quad y - 2 = -1(x - 3)$

$= \frac{7}{7} = 1$ $\quad$ so $m_{perp} = -1$ $\quad y - 2 = -x + 3$

$y + x = 5$

Now attempt Exercise 7 from the Straight Line booklet

c) $\text{Mid}_{BC} = \left(\frac{x_B+x_{BC}}{2} \ , \ \frac{y_B+y_C}{2}\right)$ $\quad m_{BC} = \frac{y_C-y_B}{x_C-x_B}$ $\quad y - b = m(x - a)$

$\text{Mid}_{BC} = \left(\frac{2+5}{2} \ , \ \frac{3+2}{2}\right)$ $\quad = \frac{2-3}{5-2}$ $\quad y - {}^5/_2 = 3(x - 4)$

$\text{Mid}_{BC} = \left(\frac{8}{2} \ , \ \frac{5}{2}\right) = \left(4\,, \frac{5}{2}\right)$ $\quad = -\frac{1}{3}$ $\quad y - {}^5/_2 = 3x - 12$

Since Perpendicular $\quad 2y - 5 = 6x - 24$

$m_{BC} \text{ x } m_{perp} = -1$ $\quad 2y - 6x = -19$

so $m_{perp} = 3$

Now attempt Exercise 8 from the Straight Line booklet

d) $5y - 3x = -5 \rightarrow ①$ $\quad\Rightarrow\quad$ $5y - 3x = -5 \rightarrow ①$

$y + x = 5 \rightarrow ②\times 3$ $\quad\Rightarrow\quad$ $3y + 3x = 15 \rightarrow ③$

$① + ③ \qquad 8y = 10$

$y = 1.25$

Sub $y = 1.25$ into ②

$1.25 + x = 5$

$x = 3.75$

POI (3.75 , 1.25)

Note:

Part **(d)** above was completed using the ELIMINATION method which is probably the one you are most comfortable with using. It could also have been solved using SUBSTITUTION as follows:

$5y - 3x = -5 \rightarrow ①$

$y + x = 5$

$y = -x + 5 \rightarrow ②$

Sub $y = -x + 5$ into ①

$\Rightarrow \quad 5(-x + 5) - 3x = -5$

$\Rightarrow \quad -5x + 25 - 3x = -5$

$\Rightarrow \quad -8x = -30$

$\Rightarrow \quad x = 3.75$

Sub $x = 3.75$ into ②

$y + 3.75 = 5$

$y = 1.25$

POI (3.75 , 1.25)

The third method of EQUATING the 2 equations would not be suitable for this question.
An example of the Equating Method is shown below.
This method is only suitable when both equations can be expressed as $y =$ or $x =$.

18. $y = 3x + 6$

$y = -2x - 4$

$3x + 6 = -2x - 4$

$5x = -10$

$x = -2$

Sub $x = -2$ into either starting equation

$y = 3(-2) + 6$

$y = 0$

POI (–2 , 0)

Now attempt Exercises 9A & 9B from the Straight Line booklet