H Rec. Relations

SPTA Mathematics – Higher Notes

- A **SEQUENCE** is an ordered list of objects, usually numbers, e.g.
  - Fibonacci Sequence: 1 , 1 , 2 , 3 , 5 , 8 , 13 , 21 …
  - Square Numbers: 1 , 4 , 9 , 16 , 25 , 36 …
  - Triangular Numbers: 1 , 3 , 6 , 10 , 15 , 21 …

- **RECURRENCE RELATIONS** are a way of describing a sequence of numbers.

- A Recurrence Relation uses the previous term to describe (or calculate) the next term in the sequence.

- To properly describe a sequence using Recurrence Relations you should always state the Initial value.

- In Higher we look at **Linear Recurrence Relations** of the form:

$$U_{n+1} = aU_n + b \qquad U_0 = c \longleftarrow \text{Initial value.}$$

Next term in the sequence.

Previous term in the sequence.

Multiplier.

Examples:

1. For the sequence described by $U_{n+1} = 3U_n - 10$, $U_0 = 15$, find U_4

$U_1 = 3U_0 - 10 = 3 \times 15 - 10 = 35$

$U_2 = 3U_1 - 10 = 3 \times 35 - 10 = 95$

$U_3 = 3U_2 - 10 = 3 \times 95 - 10 = 275$

$U_4 = 3U_3 - 10 = 3 \times 275 - 10 = \underline{815}$

2. For the sequence above find the smallest value of n for which $U_n > 20000$

$U_5 = 3U_4 - 10 = 3 \times 815 - 10 = 2435$

$U_6 = 3U_5 - 10 = 3 \times 2435 - 10 = 7295$

$U_7 = 3U_6 - 10 = 3 \times 7295 - 10 = 21875$

So the smallest value of n for which $U_n > 20000$ is $\underline{\underline{7}}$

Now attempt Exercise 1 from the Recurrence Relations booklet

3. A patient is receiving treatment in Hospital.

She receives injections of a drug to help her condition every 4 hours.

She receives the first injection of 100ml at Midday on Monday.

Every 4 hours 30% of the drug leaves the bloodstream and a further 25ml is administered.

a) Construct a Recurrence Relation to describe the amount of the drug in her system at anytime:

30% of drug leaves bloodstream so 70% left: 70% → 0.7
25ml is injected into the patient every 4 hours

This does NOT need to be written

$$U_{n+1} = 0.7U_n + 25, \quad U_0 = 100$$

b) Use the Recurrence Relation to find the amount of the drug left in the blood after 24 hours:

4pm: $U_1 = 0.7U_0 + 25 = 0.7 \times 100 + 25 = 95$

8pm: $U_2 = 0.7U_1 + 25 = 0.7 \times 95 + 25 = 91.5$

Midnight: $U_3 = 0.7U_2 + 25 = 0.7 \times 91.5 + 25 = 89.05$

4am: $U_4 = 0.7U_3 + 25 = 0.7 \times 89.05 + 25 = 87.335$

8am: $U_5 = 0.7U_4 + 25 = 0.7 \times 87.335 + 25 = 86.1345$

Midday: $U_4 = 0.7U_3 + 25 = 0.7 \times 86.1345 + 25 = 85.29415$

So after 24 hours there will be <u>85.29ml</u> of drugs in her system.

Always explain your answer.

Now attempt Exercise 2 from the Recurrence Relations booklet

Limits:

- As n tends to infinity, usually written as $n \to \infty$, the terms in a sequence will do one of 3 things:
  - Continue to get larger and larger heading towards Infinity
  - Continue to get smaller and smaller heading towards negative Infinity.
  - Converge towards a value, called a **LIMIT**.

These are called DIVERGENT SEQUENCES

- We are often asked to consider these **CONVERGENT SEQUENCES**.
- A Sequence given by $U_{n+1} = aU_n + b$, $U_0 = c$ will converge to a limit when: $-1 < a < 1$
- The limit is **NOT** dependent on the Initial Value.
- The Limit can be calculated in 2 ways , Algebraically or using the formula: $L = \frac{b}{1-a}$

Learn if going to use

- Choose the method you prefer, the formula is not given on the Formulae Sheet in the exam.
- Questions will either ask: "Calculate the Limit" or "describe what happens in the long term" or something similar.

Examples:

4. Calculate the Limit for the sequence in Example 3 above.

A limit exists since $-1 < 0.7 < 1$, Let L be the Limit.

Algebraically		**Formula**
$L = 0.7L + 25$		$L = \frac{b}{1-a}$
$0.3L = 25$	**OR**	$L = \frac{25}{1-0.7}$
$L = \frac{25}{0.3} = 83.33\text{ml}$		$L = \frac{25}{0.3} = 83.33\text{ml}$

So in the long term if the current course of treatment continues the amount of the drug will settle at but not go below 83.33ml.

For Limit Questions a statement must always be written.

5. The population of Beavers released in Scotland at the start of 2023 is 300.

The population is expected to drop by 8% each year.

To combat this fall it is decided to introduce a further 20 Beavers on the 1st of January each year.

a) How many Beavers will be in the Scotland on the 31st December 2026 years.

8% drop in Beaver population so 92% left: 92% → 0.92

$U_{n+1} = 0.92U_n + 20, \ U_0 = 300$

1st Jan 2024: $U_1 = 0.92U_0 + 20 = 0.92 \times 300 + 20 = 296$

1st Jan 2025: $U_2 = 0.92U_1 + 20 = 0.92 \times 296 + 20 = 292.32$

1st Jan 2026: $U_2 = 0.92U_1 + 20 = 0.92 \times 292.32 + 20 = 288.9344$

31st Dec 2026: $U_3 = 0.92U_2 = 0.92 \times 288.9344 = 265.82 \approx$ 265 Beavers

Don't add the 20 on here.

b) Over the long term will the population ever fall below 240 Beavers?

A limit exists since $-1 < 0.92 < 1$, Let L be the Limit.

$L = 0.92L + 20$

$0.08L = 20$

$L = \frac{20}{0.08} = 250$ Beavers

OR

$L = \frac{b}{1-a}$

$L = \frac{20}{1-0.92}$

$L = \frac{20}{0.08} = 250$ Beavers

Be very careful when asked to explain your answer.

In the long term the number of Beavers in the Scotland will increase towards but not exceed 250 Beavers.

However, note that the population **will** fall below 240 to 230 Beavers just before the 20 new animals are introduced on the 1st of January each year.

Now attempt Exercises 3A & B from the Recurrence Relations booklet

Finding a Reccurrence Relation from a sequence:

- If we know 3 consecutive entries in a sequence then we can find the linear recurrence relation.
- This process will involve **Simultaneous Equations**.

Examples:

6. A sequence is defined by $U_{n+1} = aU_n + b$, with $U_1 = 9$, $U_2 = 13$ and $U_3 = 21$.
Find a and b and then find U_6

$$U_2 = aU_1 + b \Rightarrow 13 = a \times 9 + b \quad \Rightarrow \quad 9a + b = 13 \rightarrow (1)$$

$$U_3 = aU_2 + b \Rightarrow 21 = a \times 13 + b \quad \Rightarrow \quad 13a + b = 21 \rightarrow (2)$$

$$(2) - (1) \quad \Rightarrow \quad 4a = 8$$

$$a = 2$$

$$\text{Sub } a = 2 \text{ into } (1) \quad \Rightarrow \quad 9 \times 2 + b = 13$$

$$18 + b = 13$$

$$b = -5$$

$$\text{So } U_{n+1} = 2U_n - 5 \quad \Rightarrow \quad U_4 = 2U_3 - 5 = 2 \times 21 - 5 = 37$$

$$U_5 = 2U_4 - 5 = 2 \times 37 - 5 = 69$$

$$U_6 = 2U_5 - 5 = 2 \times 69 - 5 = \underline{\underline{133}}$$

Now attempt Exercises 4 – 6 from the Recurrence Relations booklet