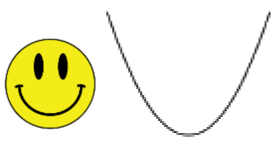
Quadratics

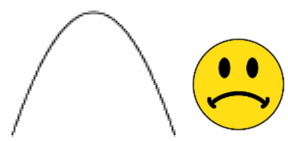
SPTA Mathematics – Higher Notes

National 5 Revision:

- Quadratics are expressions with degree 2 and are of the form $ax^2 + bx + c$, where a $\neq$ 0.
- The Graph of a Quadratic is called a Parabola, and there are 2 types as shown below:



Happy Smiley Face
Minimum Turning Point
$a > 0$

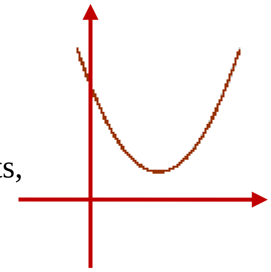


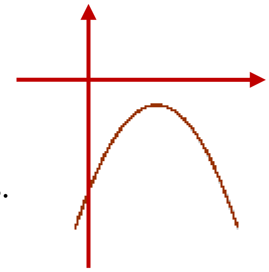
Sad Smiley Face
Maximum Turning Point
$a < 0$

- Quadratic Functions (another way of saying equation!!) can be written in 3 ways:
  - Expanded Form: $y = ax^2 + bx + c$ – This is usually the starting point for most quadratics
  - Factorised Form: $y = k(ax - b)(cx - d)$ – Useful for finding the Roots (solving)
  - Completed Square form: $y = k(ax + b)^2 + c$ – Useful for finding the Turning Point
- Solving a quadratic gives us the ROOTS of the Parabola, i.e. where it cuts the x−axis.
- There are 4 different ways to solve a quadratic equation:
  - By Factorising
  - Using the Quadratic Formula – must know this as it is **NOT** on the Formulae List.
  - From the Graph, where it crosses the x−axis.
  - From the completed square form – you may not have done this so we will go over it later!!
- We can sketch a Parabola from its equation as follows:
  - State the type of Turning Point.
  - Find the y-intercept, $x = 0$.
  - Find the roots by factorising, $y = 0$, if possible.
  - Find the coordinates of the Turning Point, midway between the roots or from completed square.
- Use the Discriminant to determine the Nature of the roots.

If you need practice of any of this try the National 5 Quadratic Revsion booklet & Videos.

Completing the Square:





- Completing the Square is a method for finding the Turning Point of a Quadratic which has no roots, i.e. it does not cross the x-axis.
  It can therefore **NOT** be factorised!!
- You saw how to Complete the Square in National 5 for simple quadratics.
- The coefficient of x^2 must be 1 before you start to Complete the Square.
- Completing the Square changes a quadratic expression $y = ax^2 + bx + c$ in to the form:
  $y = a(x + p)^2 + q$, if $a > 0$ and $y = q - a(x + p)^2$, if $a < 0$
- When in Completed Square form the Turning Point is $(-p , q)$

Examples:

National 5:

1. Write $y = x^2 - 4x + 5$ in the form $y = (x + p)^2 + q$

$$
\begin{aligned}
x^2 - 4x + 5 &= (x - 2)^2 - (2)^2 + 5 \\
&= (x - 2)^2 - 4 + 5 \\
y &= (x - 2)^2 + 1
\end{aligned}
$$

Turning Point is $(2 , 1)$

Half the coefficient of x to find p and then subtract p^2 then continue on the constant term. Now simplify!!

2. Write $y = x^2 + 3x - 4$ in the form $y = (x + p)^2 + q$

$$
\begin{aligned}
x^2 + 3x - 4 &= (x + {}^3/_2)^2 - ({}^3/_2)^2 - 4 \\
&= (x + {}^3/_2)^2 - {}^9/_4 - 4 \\
&= (x + {}^3/_2)^2 - {}^9/_4 - {}^{16}/_4 \\
y &= (x + {}^3/_2)^2 - {}^{25}/_4
\end{aligned}
$$

Turning Point is $(-{}^3/_2 , -{}^{25}/_4)$

Higher: The coefficient of x^2 must be 1 before you start to **Complete the Square**.

3. Write $y = 7 + 6x - x^2$ in the form $y = q - (x + p)^2$

$$\begin{aligned}
7 + 6x - x^2 &= 7 + 6x - x^2 \\
&= -(x^2 - 6x - 7) \\
&= -\left[(x-3)^2 - (3)^2 - 7\right] \\
&= -\left[(x-3)^2 - 9 - 7\right] \\
&= -\left[(x-3)^2 - 16\right] \\
y &= 16 - (x-3)^2
\end{aligned}$$

Rearrange and then take out a common factor of –1

Complete the square of the expression inside the brackets then multiply out and simplify.

Turning Point is (3 , 16)

4. Write $y = 3x^2 + 24x - 5$ in the form $y = a(x + p)^2 + q$

$$\begin{aligned}
3x^2 + 24x - 5 &= 3\left[x^2 + 8x\right] - 5 \\
&= 3\left[(x+4)^2 - (4)^2\right] - 5 \\
&= 3\left[(x+4)^2 - 16\right] - 5 \\
&= 3(x+4)^2 - 48 - 5 \\
y &= 3(x+4)^2 - 53
\end{aligned}$$

Take out a common factor of 3 from the first 2 terms only.

Complete the square of the expression inside the brackets then multiply out and simplify.

Turning Point is (–4 , –53)

5. Write $y = 4x^2 - 12x + 7$ in the form $y = a(x + p)^2 + q$

$$\begin{aligned}
4x^2 - 12x + 7 &= 4\left[x^2 - 3x\right] + 7 \\
&= 4\left[(x - {}^3/_2)^2 - ({}^3/_2)^2\right] + 7 \\
&= 4\left[(x - {}^3/_2)^2 - {}^9/_4\right] + 7 \\
&= 4(x - {}^3/_2)^2 - 9 + 7 \\
y &= 4(x - {}^3/_2)^2 - 2
\end{aligned}$$

Take out a common factor of 4 from the first 2 terms only.

Complete the square of the expression inside the brackets then multiply out and simplify.

Turning Point is $({}^3/_2\ ,\ -2)$

Now attempt Exercise 1 from the Quadratic Functions booklet.

Solving Quadratic by Completing the Square:

- Another way to solve a Quadratic Equation is by turning it into completed square form.
- This does not work for every equation!!

Examples:

6. Solve by first completing the square $2x^2 - 8x + 7 = 0$

$$\begin{aligned} 2x^2 - 8x + 7 &= 2\left[x^2 - 4x\right] + 7 \\ &= 2\left[(x-2)^2 - (2)^2\right] + 7 \\ &= 2\left[(x-2)^2 - 4\right] + 7 \\ &= 2(x-2)^2 - 8 + 7 \\ &= 2(x-2)^2 - 1 \end{aligned}$$

$$\begin{aligned} 2(x-2)^2 - 1 &= 0 \\ 2(x-2)^2 &= 1 \\ (x-2)^2 &= \tfrac{1}{2} \\ (x-2) &= \pm\sqrt{\tfrac{1}{2}} \\ x &= \pm \tfrac{1}{\sqrt{2}} + 2 \end{aligned}$$

or 2.71 , 1.29

Now attempt Exercise 2 from the Quadratic Functions booklet.

Solving Quadratic Inequalities:

- To solve a Quadratic Inequality you MUST do a QUICK sketch of the quadratic first but there is no need to find the coordinates of the Turning Point or the y-intercept.
- Solve it as you would a Quadratic Equation.

Examples:

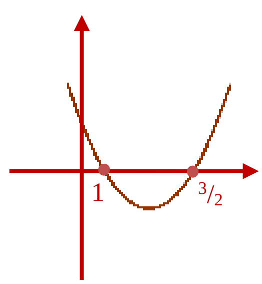
7. Solve $2x^2 - 5x + 3 > 0$

$a > 0$ so Minimum Turning Point

When $y = 0$: $2x^2 - 5x + 3 = 0$

$(2x - 3)(x - 1) = 0$

$x = 1, \, ^3/_2$



So $2x^2 - 5x + 3 > 0$, when $x < 1$ and $x > \, ^3/_2$

Note that $2x^2 - 5x + 3 < 0$ occurs when $1 < x < \, ^3/_2$

8. Solve $6 + 7x - 3x^2 > 0$

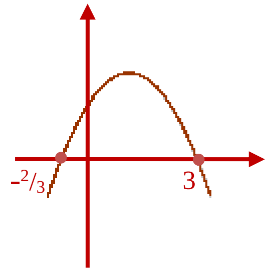
$a < 0$ so Maximum Turning Point

When $y = 0$: $\quad -3x^2 + 7x + 6 = 0$

$-(3x^2 - 7x - 6) = 0$

$-(3x + 2)(x - 3) = 0$

$x = -\frac{2}{3}, 3$



So $6 + 7x - 3x^2 > 0$, when $-\frac{2}{3} < x < 3$

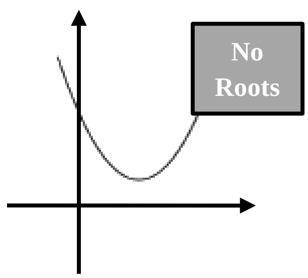
Note that $6 + 7x - 3x^2 < 0$ occurs when $x < -\frac{2}{3}$ and $x > 3$

Now attempt Exercise 3 from the Quadratic Functions booklet.

The Discriminant:

Must Know!!

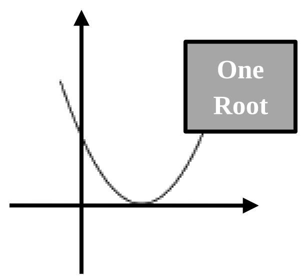
- We saw the Discriminant in National 5, $b^2 - 4ac$, part of the Quadratic Formula: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- The discriminant is used to describe the Nature of a Quadratics Roots:



$b^2 - 4ac < 0$

No Real Roots exist

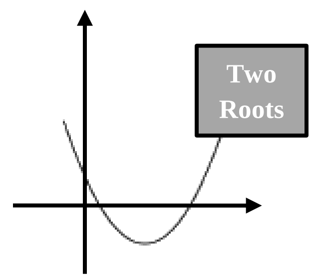
Cannot be Factorised



$b^2 - 4ac = 0$

Real & Equal Roots exist

Can be Factorised



$b^2 - 4ac > 0$

2 Real & Distinct Roots exist

May be Factorisable or may need to use the Quadratic Formula.

Note: For REAL roots, $b^2 - 4ac \geq 0$

Examples:

9. Find the Nature of the Roots of the function $f(x) = 9x^2 + 24x + 16$

$a = 9,\ b = 24,\ c = 16$

$b^2 - 4ac = 24^2 - 4(9)(16)$

$= 576 - 576$

$= 0$ since $b^2 - 4ac = 0$ there are REAL & EQUAL ROOTS.

10. Find the Nature of the Roots of the function $g(x) = -x^2 - 4x + 6$

$a = -1,\ b = -4,\ c = 6$

$$b^2 - 4ac = (-4)^2 - 4(-1)(6)$$
$$= 16 + 24$$
$$= 40 \quad \text{since } b^2 - 4ac > 0 \text{ there are 2 REAL \& DISTINCT ROOTS.}$$

11. Find the Nature of the Roots of the function $h(x) = 12 + 8x + 10x^2$

$a = 10,\ b = 8,\ c = 12$

$$b^2 - 4ac = 8^2 - 4(10)(12)$$
$$= 64 - 480$$
$$= -416 \quad \text{since } b^2 - 4ac < 0 \text{ there are No REAL ROOTS.}$$

Now attempt Exercise 4 from the Quadratic Functions booklet.

12. For what values of k does the equation $x^2 + kx + 9 = 0$ have equal roots?

$a = 1,\ b = k,\ c = 9$

Since Equal roots, $b^2 - 4ac = 0$ ← **Start with the sentence this time!!**

$$k^2 - 4(1)(9) = 0$$
$$k^2 - 36 = 0$$
$$(k - 6)(k + 6) = 0$$
$$\text{So } k = 6, k = -6$$

13. Find q given that the equation $6x^2 + 12x + q = 0$ has real roots.

$a = 6,\ b = 12,\ c = q$

Since Real roots, $b^2 - 4ac \geq 0$

$$12^2 - 4(6)(q) \geq 0$$
$$144 - 24q \geq 0$$
$$-24q \geq -144$$
$$q \leq 6$$

← **Remember to switch the sign around if multiplying or dividing by a negative.**

14. Find m such that the equation $x^2 + (mx + 3)^2 - 3 = 0$ has equal roots, $m > 0$.

$$
\begin{aligned}
x^2 + (mx+3)^2 - 3 &= x^2 + m^2x^2 + 6mx + 9 - 3 \\
&= x^2 + m^2x^2 + 6mx + 6 \\
&= (1 + m^2)x^2 + 6mx + 6
\end{aligned}
$$

$a = 1 + m^2,\ b = 6m,\ c = 6$

Since Equal roots, $b^2 - 4ac = 0$

$$
\begin{aligned}
(6m)^2 - 4(1 + m^2)(6) &= 0 \\
36m^2 - 24 - 24m^2 &= 0 \\
12m^2 - 24 &= 0 \\
m^2 - 2 &= 0 \\
(m - \sqrt{2})(m + \sqrt{2}) &= 0
\end{aligned}
$$

$m = \sqrt{2},\ \cancel{m = -\sqrt{2}}$ So $m = \sqrt{2}$

Question states $m > 0$ so this solution is not valid.

15. Show that the equation $qx^2 + px - q = 0$, $q \neq 0$, has real roots.

$a = q,\ b = p,\ c = -q$

$$
\begin{aligned}
b^2 - 4ac &= p^2 - 4(q)(-q) \\
&= p^2 + 4pq^2
\end{aligned}
$$

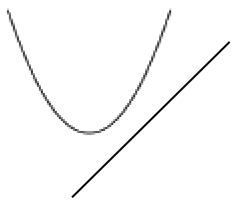
Since the sum of 2 squares can NEVER be negative, $p^2 + 4pq^2 \geq 0$ and therefore always has Real roots since $b^2 - 4ac \geq 0$

Now attempt Exercise 5 from the Quadratic Functions booklet.

Tangent to a curve:

- A Straight Line is a Tangent to a curve if it intersects (meets) the curve at one point only.
- We can use the Discriminant or Factorisation to determine if a Line is a Tangent to a curve.
  - Substitute the equation of the Line into the Equation of the Curve..
  - Use the Discriminant or Factorisation to determine how many Points of Intersection there are.
- The Discriminant can be used as follows:

$b^2 – 4ac < 0$

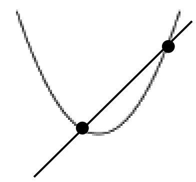


No Intersection. Not a Tangent

$b^2 – 4ac = 0$

Intersects at 1 Point. Therefore it is a Tangent

$b^2 – 4ac > 0$



Intersects at 2 Points. Not a Tangent

Examples:

16. Prove that the line $y = x – 4$ is a tangent to the curve $y = x^2 – 3x$

$x^2 – 3x = x – 4$

$x^2 – 3x – x + 4 = 0$

$x^2 – 4x + 4 = 0$

$a = 1, b = −4, c = 4$

$b^2 - 4ac = (-4)^2 - 4(1)(4)$

$= 16 - 16$

$= 0$

since $b^2 - 4ac = 0$ The line is a Tangent to the curve as there is one point of contact.

17. Determine whether or not the line $y = 5x + 7$ is a tangent to the curve $y = 3x^2 + 3x + 16$

$3x^2 + 3x + 16 = 5x + 7$

$3x^2 + 3x + 16 – 5x − 7 = 0$

$3x^2 – 2x + 9 = 0$

$a = 3, b = −2, c = 9$

$b^2 - 4ac = (-2)^2 - 4(3)(9)$

$= 4 - 108$

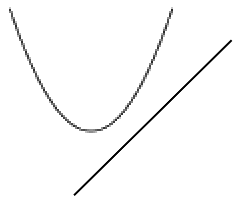
$= -104$

since $b^2 - 4ac < 0$ The line is NOT a Tangent to the curve as there are no points of contact.

Now attempt Exercise 6 from the Quadratic Functions booklet.

- Factorisation can be used as follows:

Cannot be factorised

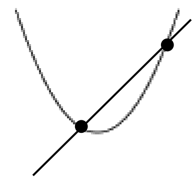


**No Intersection.
Not a Tangent**

$y = k(x - a)^2$

**Intersects at 1 Point.
Therefore it is a Tangent**

$y = k(x - a)(x - b)$



**Intersects at 2 Points.
Not a Tangent**

- Factoroisation is actually the best way to do this as often the question also asks you to find the Points of Intersection, so you will need to Factorise it anyway!!

- To find the point of contact's y–coordinate you can sub it into either equation.

Examples:

18. Same as Question 16 above.

a) Prove that the line $y = x - 4$ is a tangent to the curve $y = x^2 - 3x$

$x^2 - 3x = x - 4$

$x^2 - 3x - x + 4 = 0$

$x^2 - 4x + 4 = 0$

$(x - 2)(x - 2) = 0$

$x - 2 = 0$ & $x - 2 = 0$

So $x = 2$

Since there is only one point of contact the line is a Tangent to the curve.

b) Find the coordinates of the point of contact.

Sub $x = 2$ into $y = x - 4$: $y = (2) - 4$

$y = -2$ So Pt $(2 , -2)$

19. Determine whether or not the line $y = 4 - 5x$ is a tangent to the curve $y = 3x^2 + 6x - 16$. If possible find the points of intersection

$$3x^2 + 6x - 16 = 4 - 5x$$

$$3x^2 + 6x - 16 + 5x - 4 = 0$$

$$3x^2 + 11x - 20 = 0$$

$$(3x - 4)(x + 5) = 0$$

$$3x - 4 = 0 \quad \& \quad x + 5 = 0$$

$$3x = 4 \qquad \& \quad x = -5$$

$$x = {}^4/_3 \quad \& \quad -5$$

Since there are two points of contact the line is NOT a Tangent to the curve.

Sub $x = {}^4/_3$ into $y = 4 - 5x$: $\quad y = 4 - 5({}^4/_3)$

$y = 4 - {}^{20}/_3$

$y = -{}^8/_3$

Sub $x = -5$ into $y = 4 - 5x$: $\quad y = 4 - 5(-5)$

$y = 4 + 25$

$y = 25$ So Pts $(-5, 25)$ & $({}^4/_3, -{}^8/_3)$

Now attempt Exercise 6 & 7 from the Quadratic Functions booklet.