Integration

SPTA Mathematics – Higher Notes

- Integration is the reverse of Differentiation and is sometimes known as the Anti-Derivative.
- The symbol, $\int$, is used to represent Integration and is always used in conjunction with dx meaning "with respect to x"
- $\int ax^n \ dx = \frac{ax^{n+1}}{n+1} + c$ where c is the Constant of Integration. (Must include!)
- Integrals that include the Constant of Integration are known as **INDEFINITE INTEGRALS**.
- The Integral of a constant, k, is kx.

Examples:

1. Integrate the following:

(Should always match the variable being Integrated)

a) $\int 10x^4 \, dx$

$= \frac{10x^5}{5}$

$= 2x^5 + c$

b) $\int 5m^{-2} \, dm$

$= \frac{5m^{-1}}{-1}$

$= -5m^{-1} + c$

c) $\int 7 \, dx$

$= 7x + c$

d) $\int p^{3/2} \, dp$

$= \frac{p^{5/2}}{5/2}$

$= \frac{2p^{5/2}}{5} + c$

e) $\int (3x^4 - 4x^3 + 5) \, dx$

(You can ignore these brackets as there is nothing outside them!)

$= \frac{3x^5}{5} - \frac{4x^4}{4} + 5x$

$= \frac{3x^5}{5} - x^4 + 5x + c$

Now attempt Exercise 1 from the Integration booklet

- Prior to Integrating a function the following must be true:
  - All brackets should be multiplied out.
  - Roots need to be changed to powers: $\sqrt[m]{x^n} = x^{n/m}$
  - x cannot appear in the denominator of a fraction: $\frac{a}{bx^n} = \frac{a}{b}x^{-n}$
- After Integrating it is good practice to return the expression to the form the question gave it in.

Examples:

2. Integrate the following:

a) $\int \sqrt[3]{a^4}\, da$

$\int a^{4/3}\, da$

$= \frac{a^{7/3}}{7/3}$

$= \frac{3\sqrt[3]{a^7}}{7} + c$

c) $\int \frac{2}{3x^4} dx$

$\int \frac{2}{3}x^{-4}\, dx$

$= \frac{2}{3} \times \frac{x^{-3}}{-3}$

$= \frac{2x^{-3}}{-9}$

$= \frac{2}{-9x^3} + c$

b) $\int \frac{dx}{5x^3}$

$\int \frac{1}{5x^3}\, dx$

Alternative format but means the same.

$\int \frac{1}{5}x^{-3}\, dx$

$= \frac{1}{5} \times \frac{x^{-2}}{-2}$

$= -\frac{x^{-2}}{10} = -\frac{1}{10x^2} + c$

c) $\int \frac{2x^4 - 6x}{5} dx$

$\int \frac{2x^4}{5} - \frac{6x}{5} dx$

$= \frac{2}{5} \times \frac{x^5}{5} - \frac{6}{5} \times \frac{x^2}{2}$

$= \frac{2x^5}{25} - \frac{6x^2}{10}$

$= \frac{2x^5}{25} - \frac{3x^2}{5} + c$

d) $\int \frac{\sqrt{n}+3}{n^2} dn$

$\int \frac{n^{1/2}}{n^2} - \frac{3}{n^2} dn$

Split into separate fractions and simplify before integrating – do NOT bring the bottom up!!!!

$\int n^{-3/2} - 3n^{-2}\, dn$

$= \frac{n^{-1/2}}{-1/2} - \frac{3n^{-1}}{-1}$

$= \frac{-2}{n^{1/2}} + \frac{3}{n}$

$= \frac{-2}{\sqrt{n}} + \frac{3}{n} + c$

Now attempt Exercise 2 from the Integration booklet

Definite Integrals:

- All the above examples where Indefinite Integrals as they contained the Constant of Integration, *c*.
- A definite Integral gives a specific solution based on 2 boundary values, called Limits.
- To find a Definite Integral we do the following
  - Find the Integral as normal but miss out the + *c*
  - Sub in the Upper Limit and calculate the Integrals value.
  - Sub in the Lower Limit and calculate the Integrals value.
  - Subtract the 2 answers to find the Definite Integral value.
- This can be written as: If $F(x)$ is the integral of a function $f(x)$ then $\int_a^b f(x)\,dx = F(b) - F(a)$
- The Constant of Integration, *c*, is not included when calculating Definite Integrals as they cancel out!
- Before substituting in a value for *x*:
  - Express all negative powers as positive powers: $\frac{a}{b}x^{-n} = \frac{a}{bx^n}$
  - Express all fractional powers as roots: $ax^{n/m} = a\sqrt[m]{x^n}$

Examples:

3. Evaluate these Definite Integrals:

a) $\int_1^5 4x^2\ dx$

$= \left[\frac{4x^3}{3}\right]_1^5$

$= \frac{4(5)^3}{3} - \left(\frac{4(1)^3}{3}\right)$ ← Keep the 2nd limit in brackets incase it's negative

$= \frac{500}{3} - \left(\frac{4}{3}\right)$

$= \frac{496}{3}$

b) $\int_{-2}^0 (x^3 + 3x^2)\ dx$

$= \left[\frac{x^4}{4} + \frac{3x^3}{3}\right]_{-2}^0$

$= \left[\frac{x^4}{4} + 3x^3\right]_{-2}^0$

$= \frac{0^4}{4} + 3(0)^3 - \left(\frac{(-2)^4}{4} + 3(-2)^3\right)$

$= 0 - \left(\frac{16}{4} - 24\right)$

$= 0 - (4 - 24)$

$= 0 - (-20)$

$= 20$

c) $\int_{-1}^{3} \frac{3}{x^5}\, dx$

$\int_{-1}^{3} 3x^{-5}\, dx$

$= \left[\frac{3x^{-4}}{-4}\right]_{-1}^{3}$

$= \left[-\frac{3}{4x^4}\right]_{-1}^{3}$

$= -\frac{3}{4(3)^4} - \left(-\frac{3}{4(-1)^4}\right)$

$= -\frac{3}{324} - \left(-\frac{3}{4}\right)$

$= -\frac{1}{108} + \frac{81}{108}$

$= \frac{80}{108}$

$= \frac{20}{27}$

$= \frac{20}{27}$

4. Find the value of a: $\int_{1}^{a} (1 + 2x)\, dx = 4$

$= \left[x + \frac{2x^2}{2}\right]_{1}^{a}$

$= [x + x^2]_{1}^{a}$

$= a + a^2 - (1 + 1^2)$

$= a^2 + a - 2$

So $a^2 + a - 2 = 4$

$a^2 + a - 6 = 0$

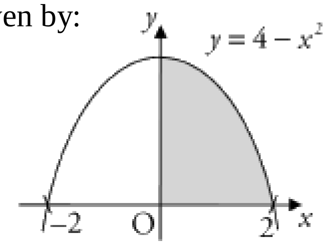
$(a + 3)(a - 2) = 0$

~~$a = -3$~~ & $a = 2$ ← **Can only be 2 as it's the upper limit and the lower limit is 1.**

So $a = 2$

Area Under a Curve:

- One interpretation of Definite Integrals is finding the Area under a Curve.
- It actually finds the Area enclosed by the Graph, between 2 Limits and the *x*-axis.
- So for the Graph opposite the Shaded Area would be given by:



$$\text{Area} = \int_0^2 (4 - x^2)\ dx$$

- If you aren't told the units then put **units²** at the end.

Examples:

5. Find the shaded area above:

$$\text{Area} = \int_0^2 (4 - x^2)\ dx$$

$$= \left[4x - \frac{x^3}{3}\right]_0^2$$

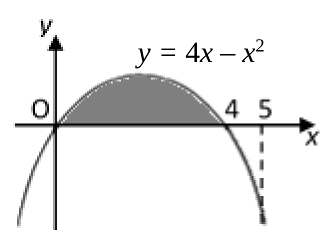
$$= 4(2) - \frac{(2)^3}{3} - \left(4(0) - \frac{(0)^3}{3}\right)$$

$$= 8 - \frac{8}{3} - (0)$$

$$= \frac{24}{3} - \frac{8}{3}$$

$$= \frac{16}{3}\ \text{units}^2$$

6. **a)** Find the shaded area shown in this graph:



$$\text{Area} = \int_0^4 (4x - x^2)\ dx$$

$$= \left[\frac{4x^2}{2} - \frac{x^3}{3}\right]_0^4$$

$$= \left[2x^2 - \frac{x^3}{3}\right]_0^4$$

$$= 2(4)^2 - \frac{4^3}{3} - \left(2(0)^2 - \frac{0^3}{3}\right)$$

$$= 32 - \frac{64}{3} - (0)$$

Now attempt Exercise 3A/B from the Integration booklet

$$= \frac{96}{3} - \frac{64}{3}$$

$$= \frac{32}{3} \text{ units}^2$$

b) For the curve above find the total area under the curve between $x = 0$ & $x = 5$:

$$\text{Area} = \left[2x^2 - \frac{x^3}{3}\right]_0^5$$

$$= 2(5)^2 - \frac{5^3}{3} - \left(2(0)^2 - \frac{0^3}{3}\right)$$

$$= 50 - \frac{125}{3} - (0)$$

$$= \frac{150}{3} - \frac{125}{3}$$

$$= \frac{25}{3} \text{ units}^2$$

How can this be smaller than the answer to part (a) if it is a bigger physical area?

Area Under a Curve – a Note of Caution!!:

- Areas above the x-axis will be Positive, and Areas below the x-axis will be Negative.
- If part of the required area is above and part below the x-axis then these areas must be calculated separately and then added, ignoring any negative signs!!
- Let's find the area under the curve above between $x = 4$ & $x = 5$:

$$\text{Area} = \left[2x^2 - \frac{x^3}{3}\right]_4^5$$

$$= 2(5)^2 - \frac{5^3}{3} - \left(2(4)^2 - \frac{4^3}{3}\right)$$

$$= 50 - \frac{125}{3} - \left(32 - \frac{64}{3}\right)$$

$$= \frac{150}{3} - \frac{125}{3} - \left(\frac{96}{3} - \frac{64}{3}\right)$$

$$= \frac{25}{3} - \frac{32}{3}$$

$$= -\frac{7}{3}$$

Since an area $= \frac{7}{3}$ units^2

Area cannot be negative so we make it positive!!

- So the total area under the curve above between $x = 0$ & $x = 5$ will be:

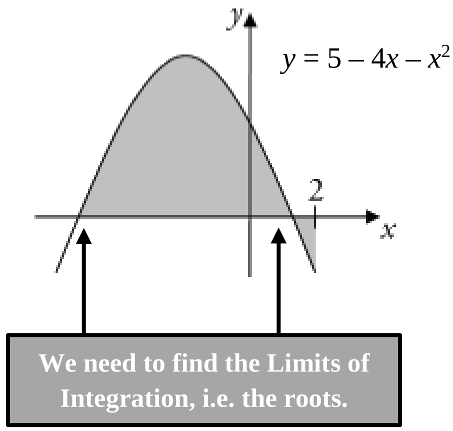
$$\text{Total area} = \frac{32}{3} + \frac{7}{3}$$

$$= \frac{39}{3}$$

$$= 13 \text{ units}^2$$

Examples:

7. Find the shaded area shown here:



Limits of Integration: $-x^2 - 4x + 5 = 0$

$$-(x^2 + 4x - 5) = 0$$

$$(x - 1)(x + 5) = 0$$

$$x - 1 = 0 \text{ or } x + 5 = 0$$

$$x = 1 \quad \& \quad x = -5$$

$$\text{Area} = \int_{-5}^{1} (-x^2 - 4x + 5)\ dx$$

$$= \left[-\frac{x^3}{3} - \frac{4x^2}{2} + 5x\right]_{-5}^{1}$$

$$= \left[-\frac{x^3}{3} - 2x^2 + 5x\right]_{-5}^{1}$$

$$= -\frac{(1)^3}{3} - 2(1)^2 + 5(1) - \left(-\frac{(-5)^3}{3} - 2(-5)^2 + 5(-5)\right)$$

$$= -\frac{1}{3} - 2 + 5 - \left(-\left(\frac{-125}{3}\right) - 50 - 25\right)$$

$$= -\frac{1}{3} + 3 - \left(\frac{125}{3} - 75\right)$$

$$= -\frac{1}{3} + \frac{9}{3} - \left(\frac{125}{3} - \frac{225}{3}\right)$$

$$= \frac{8}{3} - \left(-\frac{100}{3}\right) = \frac{108}{3} = 36 \text{ units}^2$$

$$\text{Area} = \left[-\frac{x^3}{3} - 2x^2 + 5x\right]_{1}^{2}$$

$$= -\frac{(2)^3}{3} - 2(2)^2 + 5(2) - \left(-\frac{(1)^3}{3} - 2(1)^2 + 5(1)\right)$$

$$= -\frac{8}{3} - 8 + 10 - \left(-\frac{1}{3} - 2 + 5\right)$$

$$= -\frac{8}{3} + 2 - \left(-\frac{1}{3} + 3\right)$$

$$= -\frac{8}{3} + \frac{6}{3} - \left(-\frac{1}{3} + \frac{9}{3}\right)$$

$$= -\frac{2}{3} - \frac{8}{3}$$

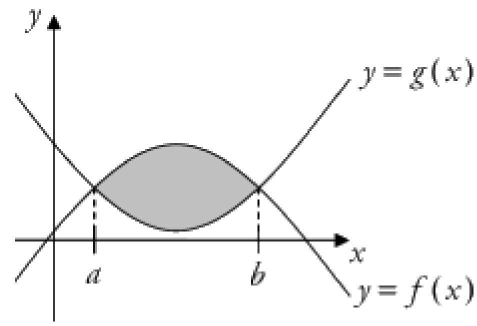
$$= -\frac{10}{3}$$

$$= \frac{10}{3} \text{ units}^2$$

$$\text{Total Area} = \frac{10}{3} + \frac{108}{3} = \frac{118}{3} \text{ or } 39\frac{1}{3} \text{ units}^2$$

Now attempt Exercise 4 from the Integration booklet

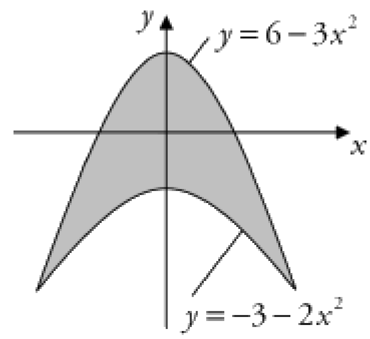
Area Between 2 Curves:



- The Area between 2 Curves is given by:
  $\int_a^b f(x) - g(x)\,dx$ or $\int_a^b (upper\ curve - lower\ curve)\,dx$
- The Limits, *a* and *b*, are normally the points of intersection of the 2 curves found by equating the 2 curves, i.e. $f(x) = g(x)$
- You may need to use Synthetic Division to find the Limits.
- Areas above/below the *x*-axis do **NOT** need to be found separately here!
- If the curves swap positions (upper becomes lower & vice versa) then these areas need to be found separately and then added together.

Examples:

8. Find the shaded area shown here:



Limits of Integration: $6 - 3x^2 = -3 - 2x^2$

$$9 - x^2 = 0$$

$$(3 - x)(3 + x) = 0$$

$$3 - x = 0 \text{ or } 3 + x = 0$$

$$x = 3 \quad \& \quad x = -3$$

Area $= \int_{-3}^{3} 9 - x^2\ dx$ ← **Use first expression equal to zero above**

$$= \left[9x - \frac{x^3}{3}\right]_{-3}^{3}$$

$$= 9(3) - \frac{(3)^3}{3} - \left(9(-3) - \frac{(-3)^3}{3}\right)$$

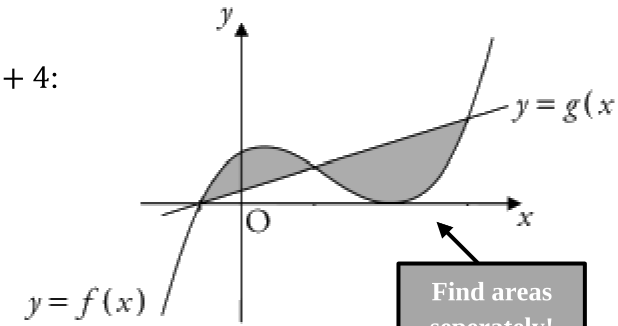
$$= 27 - \frac{27}{3} - \left(-27 - \frac{-27}{3}\right)$$

$$= 27 - 9 - (-27 + 9)$$

$$= 18 - (-18)$$

$$= 36 \text{ units}^2$$

9. Find the shaded area enclosed by $f(x) = x^3 - 7x^2 + 8x + 16$ and $g(x) = 4x + 4$:



Find areas seperately!

Limits of Integration:

$x^3 - 7x^2 + 8x + 16 = 4x + 4$

$x^3 - 7x^2 + 4x + 12 = 0$

Synthetic Division to solve a cubic with factors of 12

2	1	–7	4	12
	↓	2	–10	–12
	1	–5	–6	0

Since Remainder = 0 $(x - 2)$ is a factor.

Hence, $x^3 - 7x^2 + 4x + 12 = 0$

$\Longrightarrow (x-2)(x^2 - 5x - 6) = 0$

$\Longrightarrow (x-2)(x-6)(x+1) = 0$

$\Longrightarrow x = -1, 2, 6$

$$\text{Area} = \int_{-1}^{2} x^3 - 7x^2 + 8x + 16 - (4x+4)\ dx$$

$$= \int_{-1}^{2} x^3 - 7x^2 + 4x + 12\ dx$$

$$= \left[\frac{x^4}{4} - \frac{7x^3}{3} + 2x^2 + 12x\right]_{-1}^{2}$$

$$= \frac{(2)^4}{4} - \frac{7(2)^3}{3} + 2(2)^2 + 12(2) - \left(\frac{(-1)^4}{4} - \frac{7(-1)^3}{3} + 2(-1)^2 + 12(-1)\right)$$

$$= \frac{16}{4} - \frac{56}{3} + 8 + 24 - \left(\frac{1}{4} - \frac{-7}{3} + 2 - 12\right)$$

$$= 36 - \frac{56}{3} - \left(\frac{1}{4} + \frac{7}{3} - 10\right)$$

$$= 46 - \frac{63}{3} - \frac{1}{4}$$

$$= 46 - 21 - \frac{1}{4}$$

$$= 25 - \frac{1}{4}$$

$$= \frac{100}{4} - \frac{1}{4}$$

$$= \frac{99}{4}$$

$$= 24\frac{3}{4}\ \text{units}^2$$

$$\text{Area} = \int_2^6 4x + 4 - (x^3 - 7x^2 + 8x + 16)\ dx$$

$$= \int_2^6 -x^3 + 7x^2 - 4x - 12\ dx$$

$$= \left[-\frac{x^4}{4} + \frac{7x^3}{3} - 2x^2 - 12x\right]_2^6$$

$$= -\frac{(6)^4}{4} + \frac{7(6)^3}{3} - 2(6)^2 - 12(6) - \left(-\frac{(2)^4}{4} + \frac{7(2)^3}{3} - 2(2)^2 - 12(2)\right)$$

$$= \frac{-1296}{4} + \frac{1512}{3} - 72 - 72 - \left(-4 + \frac{56}{3} - 8 - 24\right)$$

$$= -324 + 504 - 144 - \left(-4 + \frac{56}{3} - 32\right)$$

$$= 72 - \frac{56}{3}$$

$$= \frac{216}{3} - \frac{56}{3}$$

$$= \frac{160}{3} = 53\frac{1}{3}\ \text{units}^2$$

$$\text{Total Area} = \frac{160}{3} + \frac{99}{4}$$

$$= \frac{640}{12} + \frac{297}{12}$$

$$= \frac{937}{12}$$

$$= 78\frac{1}{12}\ \text{units}^2$$

10. A farmer is thinking of buying a new water trough for his cows.

His old trough holds 600 litres of water and he wants a new one which will hold at least this amount of water.

The trough is 3 metres long and the cross-section is made up of a Parabola, $y = x^2 - 7x + 11$ cut off by the line $y = 5$.

1 unit = 10cm.

Should the farmer buy this new trough?

Limits of Integration:

$$5 = x^2 - 7x + 11$$

$$-x^2 + 7x - 6 = 0$$

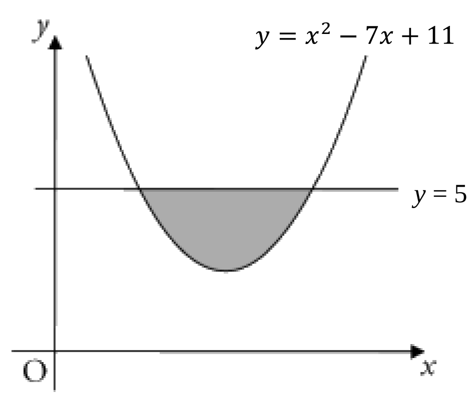
$$-(x^2 - 7x + 6) = 0$$

$$(x-1)(x-6) = 0$$

$$x - 1 = 0 \text{ or } x - 6 = 0$$

$$x = 1 \text{ or } x = 6$$

Use this expression as the Integral



$$\text{Area} = \int_1^6 -x^2 + 7x - 6 \; dx$$

$$= \left[-\frac{x^3}{3} + \frac{7}{2}x^2 - 6x\right]_1^6$$

$$= -\frac{(6)^3}{3} + \frac{7}{2}(6)^2 - 6(6) - \left(-\frac{(1)^3}{3} + \frac{7}{2}(1)^2 - 6(1)\right)$$

$$= -\frac{216}{3} + 126 - 36 - \left(-\frac{1}{3} + \frac{7}{2} - 6\right)$$

$$= -72 + 90 - \left(-\frac{2}{6} + \frac{21}{6} - \frac{36}{6}\right)$$

$$= 18 - \left(-\frac{17}{6}\right)$$

$$= \frac{108}{6} - \left(-\frac{17}{6}\right)$$

$$= \frac{125}{6}$$

$= 20\frac{5}{6}$ units^2 $\qquad$ 1 unit $=$ 10cm so 1 unit^2 $=$ 100cm^2

$$= 20\frac{5}{6} \times 100 \text{ cm}^2$$

$$= 2083\frac{1}{3} \text{ cm}^2$$

So Volume $= 2083\frac{1}{3} \times 300$

$= 625000$ cm^3 $\qquad$ 1cm^3 $=$ 1ml so 1000cm^3 $=$ 1 Litre

$= 625$ Litres

So the farmer should buy this trough as $625 > 600$ Litres.

Now attempt Exercise 5 from the Integration booklet

Differential Equations:

- Remember that Integration is the reverse of Differentiation.
- A Differential Equation is any equation which contains $\frac{dy}{dx}$ or $f'(x)$ e.g. $\frac{dy}{dx} = 4x - 5$
- In Higher we only consider first order differential equations.
- To solve a Differential Equation we Integrate to find the General Solution which involves $+c$.
- With more information, such as a point we can find the Particular Solution not involving c.

Examples:

11. a) Find the Particular solution for the differential equation above if it passes through (2 , 5):

$\frac{dy}{dx} = 3x - 5 \quad \Rightarrow y = \int 4x - 5\, dx$

$\Rightarrow y = 2x^2 - 5x + c$ ← **General Solution**

When $x = 2, y = 5 \quad \Rightarrow 5 = 2(2)^2 - 5(2) + c$

$\Rightarrow 5 = 8 - 10 + c$

$\Rightarrow 5 = -2 + c$

$\Rightarrow c = 7$

$\Rightarrow y = 2x^2 - 5x + 7$ ← **Particular Solution**

b) A function, f, defined on a suitable domain such that $f'(x) = x^3 - \frac{1}{x^2} + 3$.
Given that $f(2) = 7$ find $f(x)$ in terms of x

$f'(x) = x^3 - \frac{1}{x^2} + 3 \quad \Rightarrow f(x) = \int x^3 - \frac{1}{x^2} + 3\, dx$

$\Rightarrow f(x) = \int x^3 - x^{-2} + 3\, dx$

$\Rightarrow f(x) = \frac{x^4}{4} + x^{-1} + 3x + c$

$\Rightarrow f(x) = \frac{x^4}{4} + \frac{1}{x} + 3x + c$

When $x = 2, f(2) = 7 \quad \Rightarrow 7 = \frac{(2)^4}{4} + \frac{1}{2} + 3(2) + c$

$\Rightarrow 7 = 4 + \frac{1}{2} + 6 + c$

$\Rightarrow 7 = 10\frac{1}{2} + c$

$\Rightarrow c = -3\frac{1}{2} \Rightarrow f(x) = \frac{x^4}{4} + \frac{1}{x} + 3x - 3\frac{1}{2}$

Now attempt Exercise 6 – 8 from the Integration booklet