Graphs of Functions

SPTA Mathematics – Higher Notes

Transformations:

- There are 3 types of **TRANSORMATIONS** which can be applied to any Graph:.
  - A **TRANSLATION** moves every point on a graph the same amount in a specific direction.
  - A **REFLECTION** "flips" the graph about one of the axis.
  - A **SCALING** stretches or compresses the graph in a specific direction.
- Each transformation can be applied vertically or horizontally depending where it's stated in the function:
  - If the transformation is **INSIDE** the bracket it is a **HORIZONTAL** movement and affects the graph in the **OPPOSITE** manner than first thought!!!!
  - If the transformation is **OUTSIDE** the bracket it is a **VERTICAL** movement and affects the graph in the normal way!!!

Translations:

- There are 2 types of **TRANSLATIONS** which can be applied to a Graph:
  - Horizontally: $f(x + a)$ This will cause the graph to move parallel to the x-axis; to the **LEFT** if $\boldsymbol{a} > 0$ and **RIGHT** if $\boldsymbol{a} < 0$ Only the x-coordinate will change.

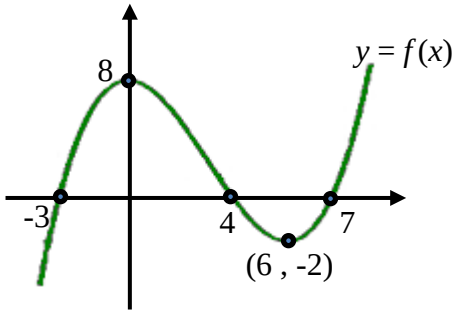
    Opposite from the way you think!!
  - Vertically: $f(x) + a$ This will cause the graph to move parallel to the y-axis; **UP** if $\boldsymbol{a} > 0$ and **DOWN** if $\boldsymbol{a} < 0$ Only the y-coordinate will change.

Examples:

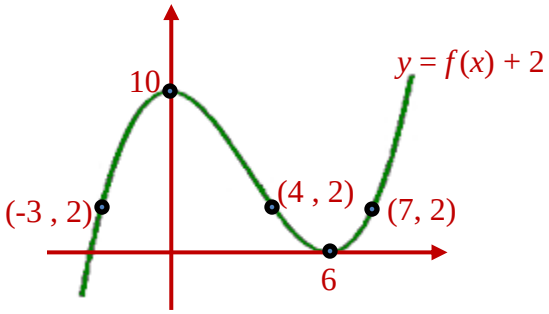
1. Sketch the following graphs:

a) $y = f(x) + 2$

Outside the brackets so shift the Graph **UP** 2



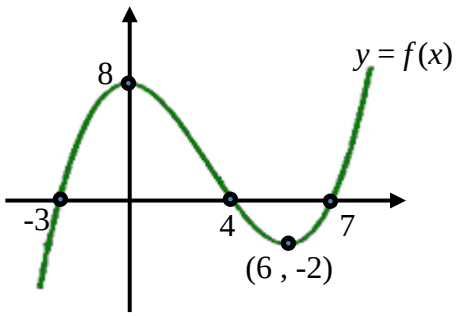
(−3 , 0)	→	(−3 , 2)
(0 , 8)	→	(0 , 10)
(4 , 0)	→	(4 , 2)
(6 , −2)	→	(6 , 0)
(7 , 0)	→	(7 , 2)



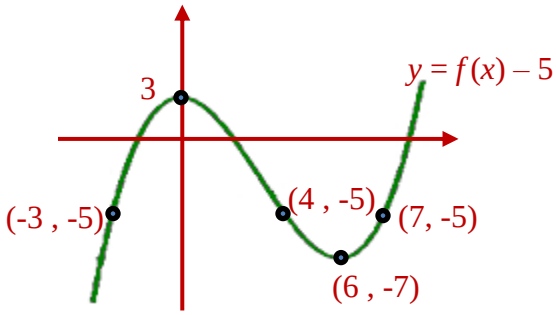
You must annotate the graph by marking in ALL the given points on the new graph to gain full marks!!!

b) $y = f(x) - 5$

Outside the brackets so shift the Graph **DOWN** 5

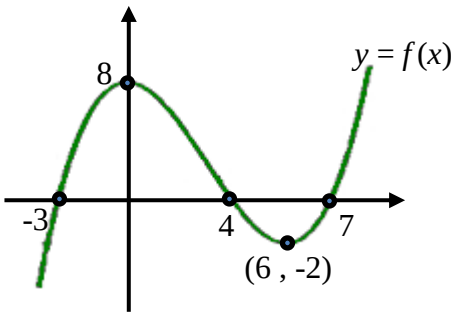


(−3 , 0)	→	(−3 , −5)
(0 , 8)	→	(0 , 3)
(4 , 0)	→	(4 , −5)
(6 , −2)	→	(6 , −7)
(7 , 0)	→	(7 , −5)



Now attempt Exercise 1 from the Graphs of Functions booklet

2. Sketch the following graphs:

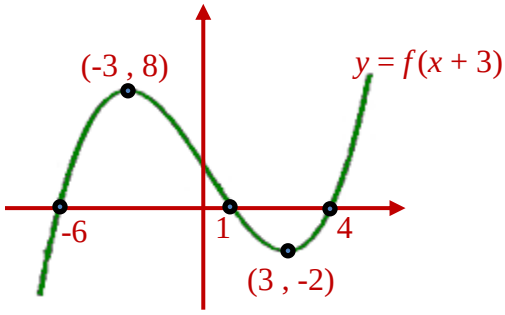


a) $y = f(x + 3)$

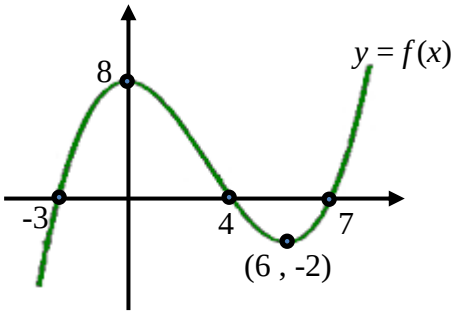
Inside the brackets so shift the Graph **LEFT** 3

Do the OPPOSITE!!!

(−3 , 0)	→	(−6, 0)
(0 , 8)	→	(−3 , 8)
(4 , 0)	→	(1 , 0)
(6 , −2)	→	(3 , −2)
(7 , 0)	→	(4 , 0)



Remember to annotate the graph by marking in ALL the given points on the new graph to gain full marks!!!

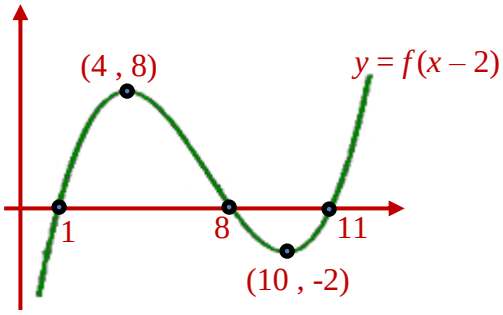


b) $y = f(x - 4)$

Inside the brackets so shift the Graph **RIGHT** 4

Do the OPPOSITE!!!

(−3 , 0)	→	(1, 0)
(0 , 8)	→	(4 , 8)
(4 , 0)	→	(8 , 0)
(6 , −2)	→	(10 , −2)
(7 , 0)	→	(11 , 0)



Now attempt Exercise 2 from the Graphs of Functions booklet

Reflections:

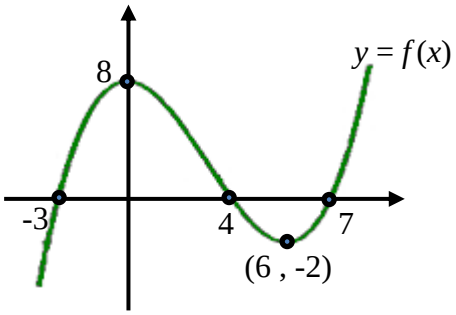
- There are 2 types of **REFLECTIONS** which can be applied to a Graph:.
  - x – axis: $-f(x)$ Reflects the graph in the x-axis.
    Changes the signs of all the y-coordinates, x stays the same!
  - y – axis: $f(-x)$ Reflects the graph in the y-axis.
    Changes the signs of all the x-coordinates, y stays the same!

Examples:

3. a) Sketch the following graphs:

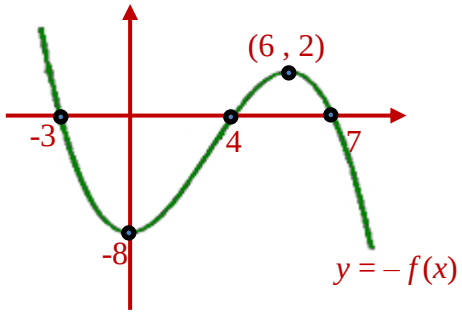
$y = -f(x)$

Outside the bracket so reflect the Graph in the ***x*-axis**



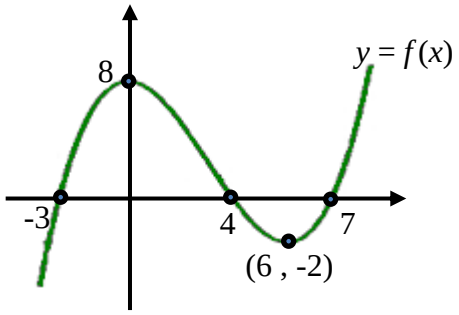
A point (a , b) will become (a , –b) when reflected in the x-axis. Points on the *x*-axis stay the same.

(−3 , 0)	→	(−3, 0)
(0 , 8)	→	(0 , −8)
(4 , 0)	→	(4 , 0)
(6 , −2)	→	(6 , 2)
(7 , 0)	→	(7 , 0)



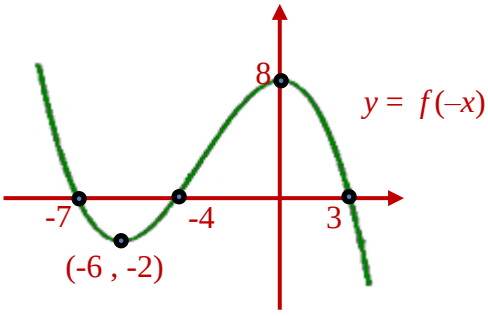
b) $y = f(-x)$

Inside the bracket so reflect the Graph in the ***y*-axis**



A point (a , b) will become (–a , b) when reflected in the y-axis. Points on the *y*-axis stay the same.

(−3 , 0)	→	(3 , 0)
(0 , 8)	→	(0 , 8)
(4 , 0)	→	(−4 , 0)
(6 , −2)	→	(−6 , −2)
(7 , 0)	→	(−7 , 0)



Now attempt Exercise 3 from the Graphs of Functions booklet

Scaling:

- There are 2 types of **SCALINGS** which can be applied to a Graph:.

  - Horizontally: $f(kx)$ This will cause the **WIDTH** of the graph to be changed as follows:

    Opposite from the way you thimk!! →

    STRETCHED if $0 < k < 1$ or **NARROWED** if $k > 1$

    Only the x-coordinate will change by dividing it by k.

  - Vertically: $k\,f(x)$ This will cause the **HEIGHT** of the graph to be changed as follows:

    STRETCHED if $k > 1$ or **COMPRESSED** if $0 < k < 1$

    Only the y-coordinate will change by multiplying it by k.

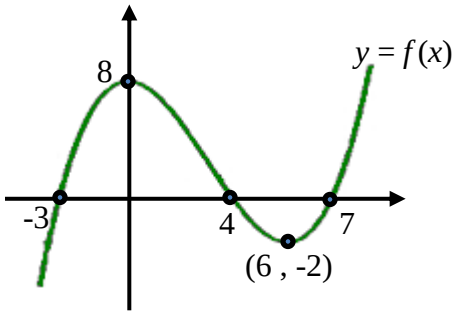
Examples:

4. Sketch the following graphs:

a) $y = f(2x)$

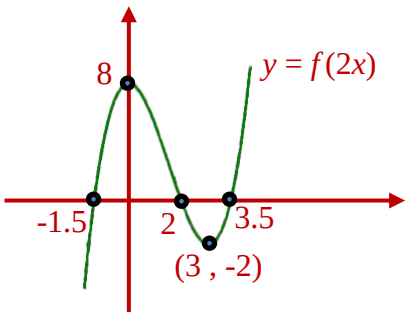
Inside the bracket so **NARROW** the Graph horizontally by a factor of 2

Do the OPPOSITE!!!



$(-3\ ,\ 0)$	→	$(-1.5\ ,\ 0)$
$(0\ ,\ 8)$	→	$(0\ ,\ 8)$
$(4\ ,\ 0)$	→	$(2\ ,\ 0)$
$(6\ ,\ -2)$	→	$(3\ ,\ -2)$
$(7\ ,\ 0)$	→	$(3.5\ ,\ 0)$

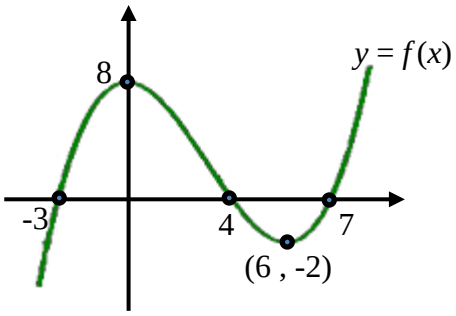
The y-coordinate stays the same!!



b) $y = f(\frac{1}{2}x)$

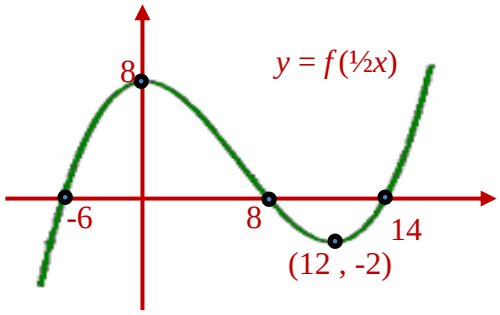
Inside the bracket so **STRETCH** the Graph horizontally by a factor of 2

Do the OPPOSITE!!!



(−3 , 0)	→	(−6 , 0)
(0 , 8)	→	(0 , 8)
(4 , 0)	→	(8 , 0)
(6 , −2)	→	(12 , −2)
(7 , 0)	→	(14 , 0)

The *y*-coordinate stays the same!!

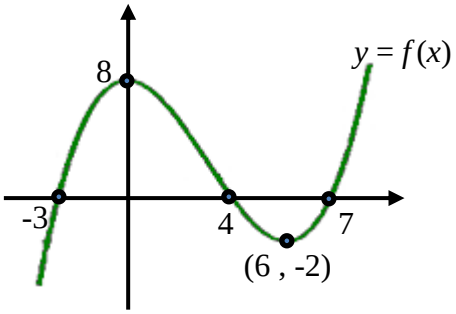


Now attempt Exercise 4 from the Graphs of Functions booklet

5. Sketch the following graphs:

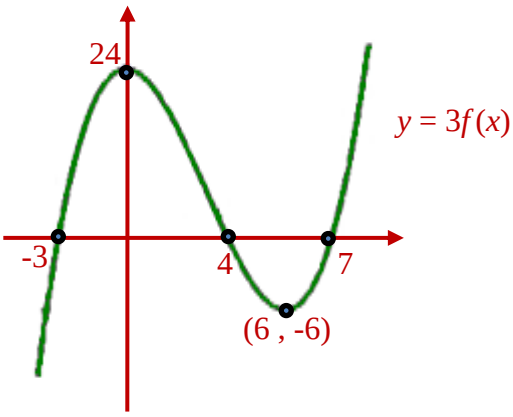
a) $y = 3f(x)$

Outside the bracket so **STRETCH** the Graph vertically by a factor of 3



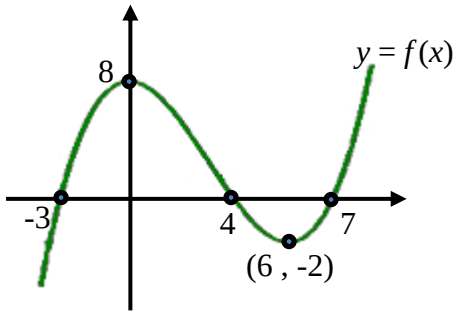
(−3 , 0)	→	(−6 , 0)
(0 , 8)	→	(0 , 24)
(4 , 0)	→	(4 , 0)
(6 , −2)	→	(6 , −6)
(7 , 0)	→	(7 , 0)

The *x*-coordinate stays the same!!



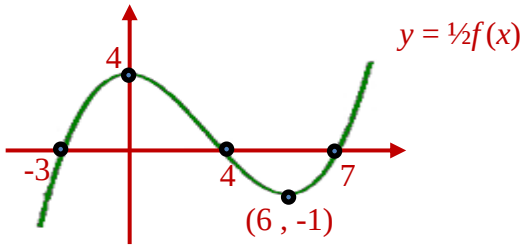
b) $y = \frac{1}{2}f(x)$

Outside the bracket so **COMPRESS** the Graph vertically by a factor of 2



(−3 , 0)	→	(−6 , 0)
(0 , 8)	→	(0 , 4)
(4 , 0)	→	(4 , 0)
(6 , −2)	→	(6 , −1)
(7 , 0)	→	(7 , 0)

The *x*-coordinate stays the same!!



Now attempt Exercise 5 from the Graphs of Functions booklet

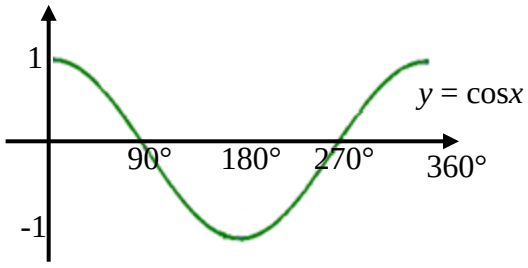
Examples - Related Trig Graphs:

- Remember **Sine, Cosine & Tangent** graphs we can apply the transformations looked at above to produce related Trig graphs in the same way.
- Trig graphs will usually be sketched between zero and 360°

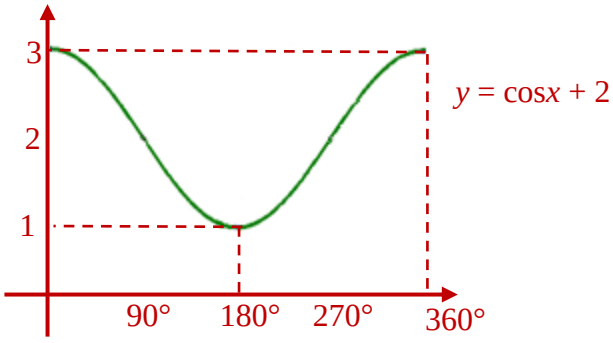
6. Sketch the following graphs:

a) $y = \cos x + 2$

Outside the brackets so shift the Graph **UP** 2



(0 , 1)	→	(0 , 3)
(90 , 0)	→	(90 , 2)
(180 , −1)	→	(180 , 1)
(270 , 0)	→	(270 , 2)
(360 , 1)	→	(360 , 3)

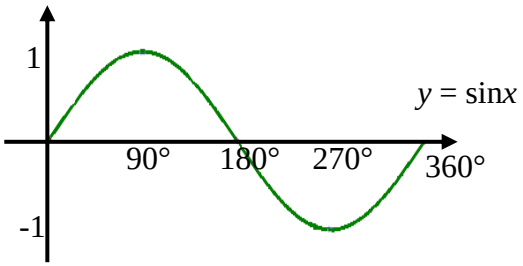


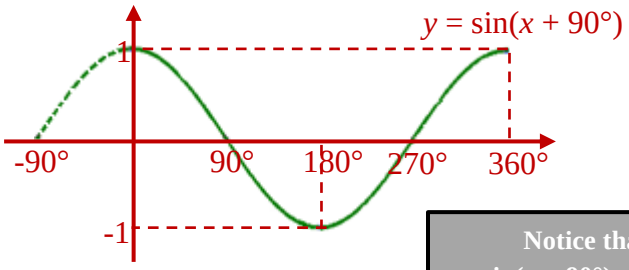
b) $y = \sin(x + 90°)$

Inside the brackets so shift the Graph **LEFT** 90°

Do the OPPOSITE!!!

(0 , 0)	→	(−90 , 0)
(90 , 1)	→	(0 , 1)
(180 , 0)	→	(90 , 0)
(270 , −1)	→	(180 , −1)
(360 , 0)	→	(270 , 0)



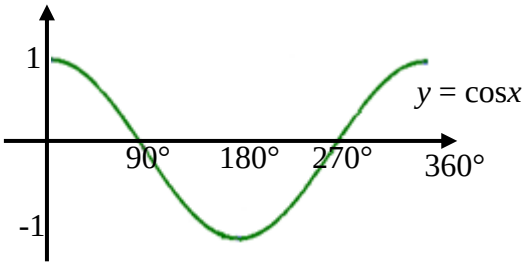


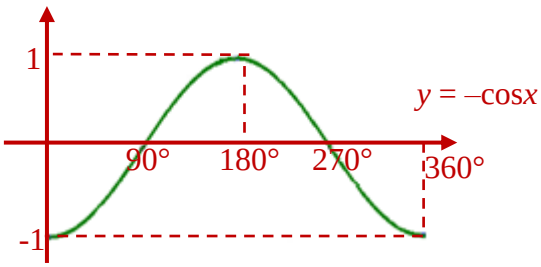
Notice that
sin(x + 90°) = cosx

c) $y = -\cos x$

Outside the brackets so reflect the Graph ***x*−axis**

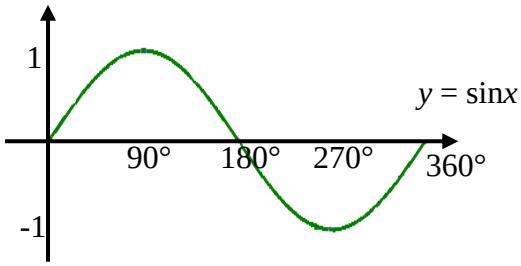
(0 , 1)	→	(0 , −1)
(90 , 0)	→	(90 , 0)
(180 , −1)	→	(180 , 1)
(270 , 0)	→	(270 , 0)
(360 , 1)	→	(360 , −1)



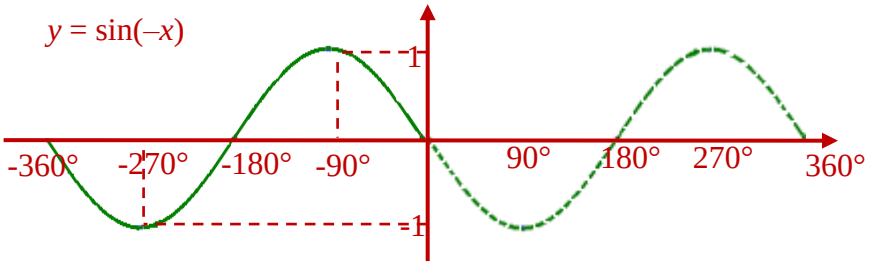


d) $y = \sin(-x)$

Inside the bracket so reflect the Graph in the ***y*-axis**



(0 , 0)	→	(0 , 0)
(90 , 1)	→	(−90 , 1)
(180 , 0)	→	(−180 , 0)
(270 , −1)	→	(−270 , −1)
(360 , 0)	→	(−360 , 0)



Notice that the graph of y = sin(–*x*) above would be the same as the graph of y = –sin*x* (reflected in *x*–axis. This means that we can write sin(–*x*) = –sin*x*. We will see more of this later in the course.

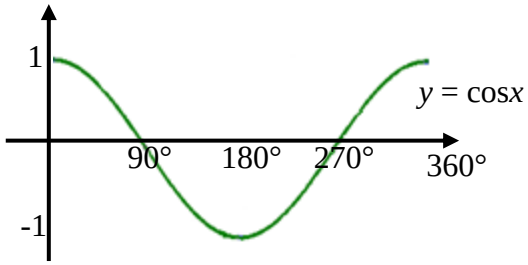
e) $y = \cos 2x$

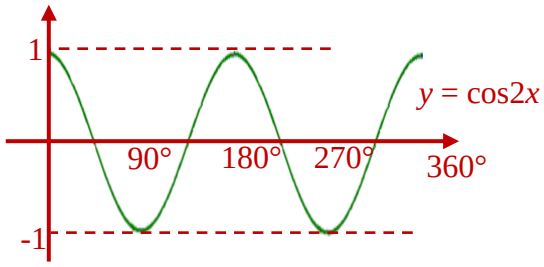
Think of it as cos(2*x*)

NARROW the Graph horizontally by a factor of 2

Do the OPPOSITE!!!

(0 , 1)	→	(0 , 1)
(90 , 0)	→	(45 , 0)
(180 , −1)	→	(90 , −1)
(270 , 0)	→	(135 , 0)
(360 , 1)	→	(180 , 1)





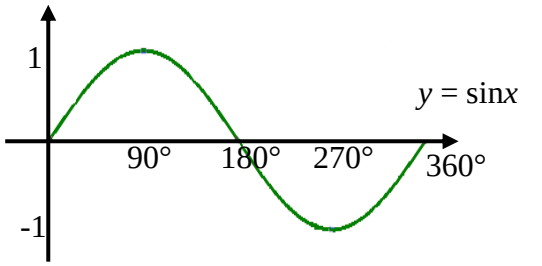
f) $y = \sin\frac{1}{2}x$

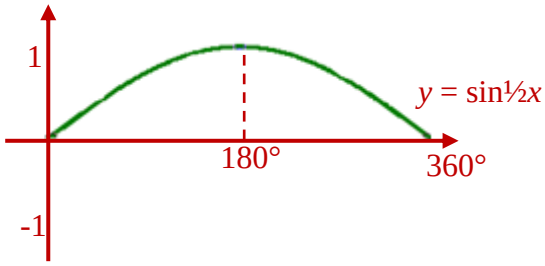
Think of it as $\sin(\frac{1}{2}x)$

STRETCH the Graph horizontally by a factor of 2

Do the OPPOSITE!!!

(0 , 0)	→	(0 , 0)
(90 , 1)	→	(180 , 1)
(180 , 0)	→	(360 , 0)
(270 , −1)	→	(540 , −1)
(360 , 0)	→	(720 , 0)

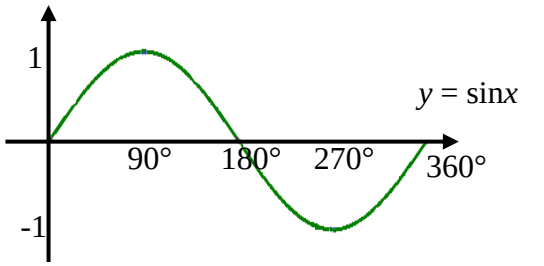


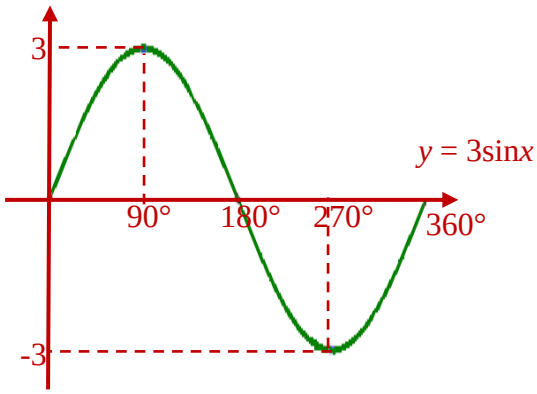


g) $y = 3\sin x$

STRETCH the Graph vertically by a factor of 3

(0 , 0)	→	(0 , 0)
(90 , 1)	→	(90 , 3)
(180 , 0)	→	(180 , 0)
(270 , −1)	→	(270 , −3)
(360 , 0)	→	(360 , 0)





Now attempt Exercise 6 from the Graphs of Functions booklet

Examples - Combined Questions:

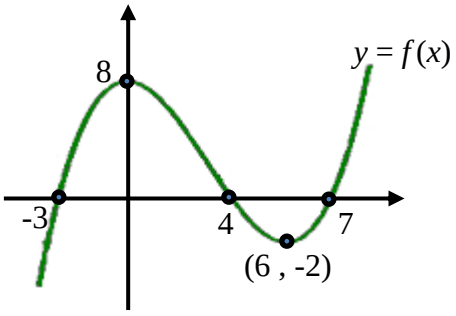
- Usually you are asked to sketch a graph with 2 (or more) transformations.
- Do any Scaling and Reflections first then do the Translations.

7. Sketch the following graphs:

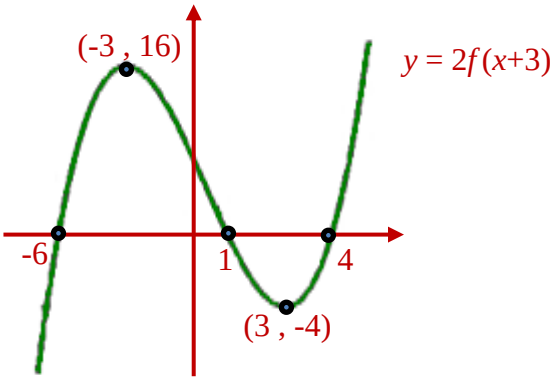
a) $y = 2f(x+3)$

STRETCH the Graph vertically by a factor of 2 then shift the Graph **LEFT** 3

Do the OPPOSITE!!!



(−3 , 0)	→	(−3 , 0)	→	(−6 , 0)
(0 , 8)	→	(0 , 16)	→	(−3 , 16)
(4 , 0)	→	(4 , 0)	→	(1 , 0)
(6 , −2)	→	(6 , −4)	→	(3 , −4)
(7 , 0)	→	(7 , 0)	→	(4 , 0)

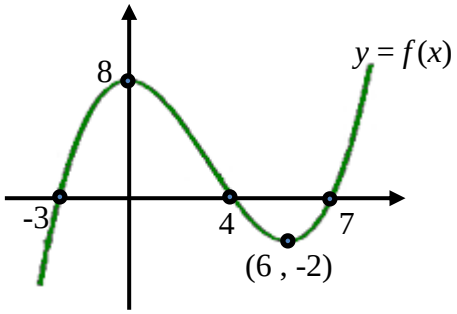


b) $y = 2 - f(x)$

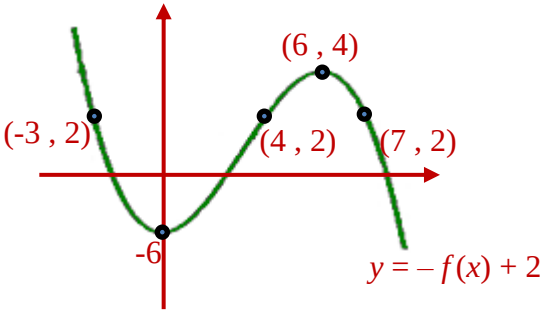
$y = -f(x) + 2$

Always rearrange so that *f* (*x*) is first

FLIP the Graph vertically then shift the Graph **UP** 2

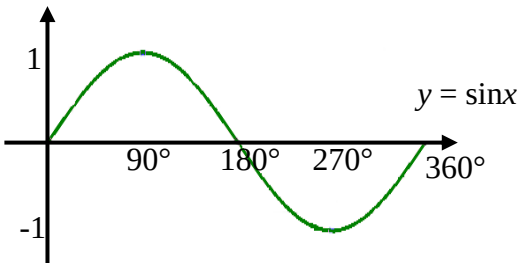


(−3 , 0)	→	(−3 , 0)	→	(−3 , 2)
(0 , 8)	→	(0 , −8)	→	(0 , −6)
(4 , 0)	→	(4 , 0)	→	(4 , 2)
(6 , −2)	→	(6 , 2)	→	(6 , 4)
(7 , 0)	→	(7 , 0)	→	(7 , 2)

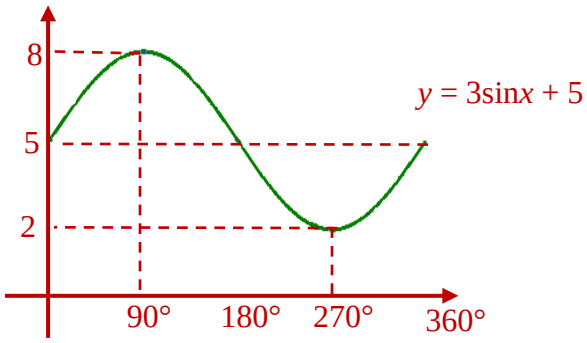


c) $y = 3\sin x + 5$

STRETCH the Graph vertically then shift the Graph **UP** 5

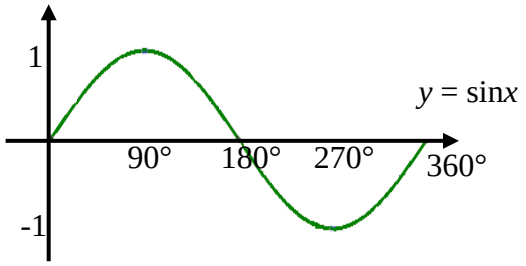


(0 , 0)	→	(0 , 0)	→	(0 , 5)
(90 , 1)	→	(90 , 3)	→	(90 , 8)
(180 , 0)	→	(180 , 0)	→	(180 , 5)
(270 , −1)	→	(270 , −3)	→	(270 , 2)
(360 , 0)	→	(360 , 0)	→	(360 , 5)

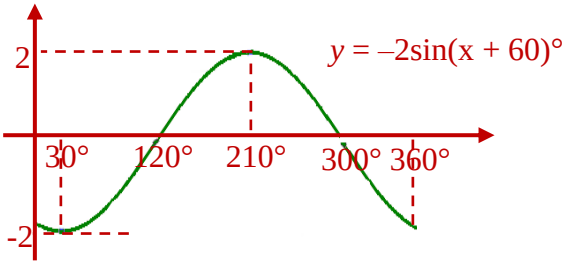


d) $y = -2\sin(x + 60)°$

FLIP & STRETCH the Graph vertically then shift the Graph **LEFT** 60°



(0 , 0)	→	(0 , 0)	→	(−60 , 0)
(90 , 1)	→	(90 , −2)	→	(30 , 8)
(180 , 0)	→	(180 , 0)	→	(120 , 0)
(270 , −1)	→	(270 , 2)	→	(210 , 2)
(360 , 0)	→	(360 , 0)	→	(300 , 0)



Now attempt Exercise 7 from the Graphs of Functions booklet

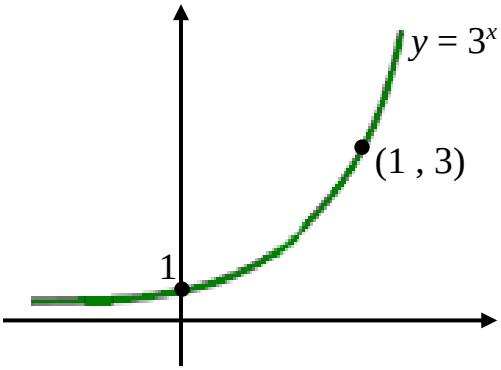
Examples - Related Exponential & Logarithmic Graphs:

- Remember **EXPONENTIAL** graphs, $y = a^x$, always pass through the points: (0 , 1) & (1 , a)

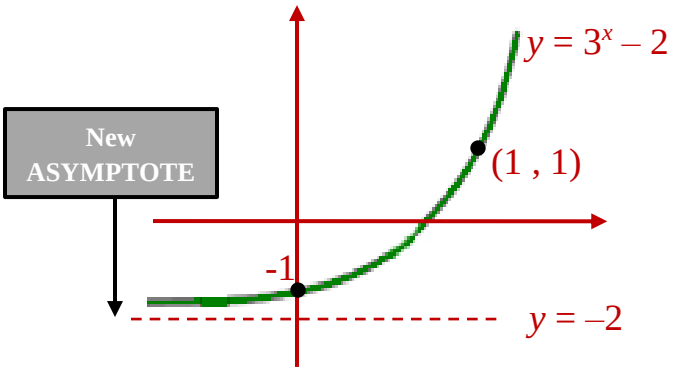
8. The graph of $y = 3^x$, is shown below, sketch the graphs of:

a) $y = 3^x - 2$

b) $y = 3^{2x}$



a) Shift the Graph **DOWN** 2

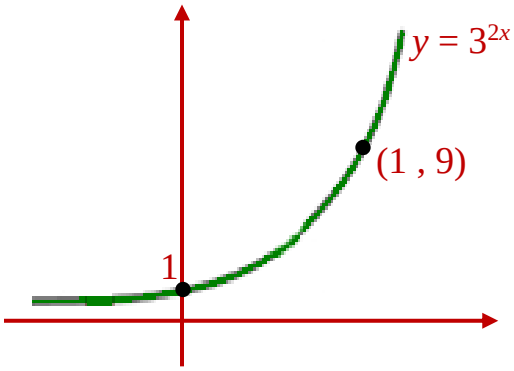


(0 , 1) → (0 , −1)

(1 , 3) → (1 , 1)

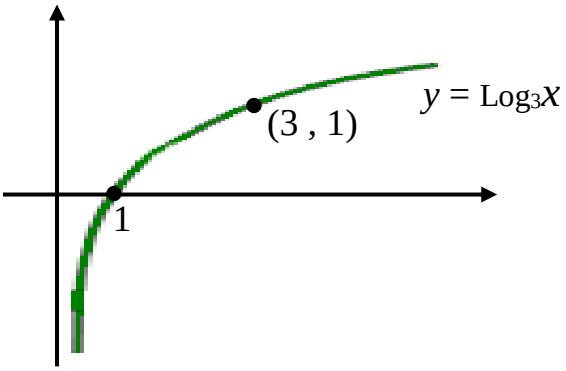
(b) $y = 3^{2x} = (3^2)^x = 9^x$

Graph passes through (0 , 1) & (1 , 9)



- Remember **LOGARITHMIC** graphs, $y = \text{Log}_a x$, always pass through the points: $(1 , 0)$ & $(a , 1)$

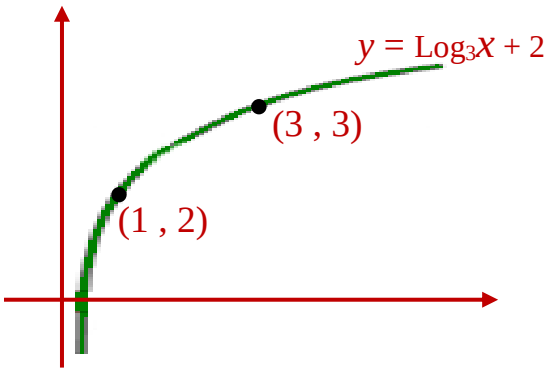
9. The graph of $y = \text{Log}_3 x$, is shown below, sketch the graphs of:



a) $y = \text{Log}_3 x + 2$

b) $y = \text{Log}_3(x - 5)$

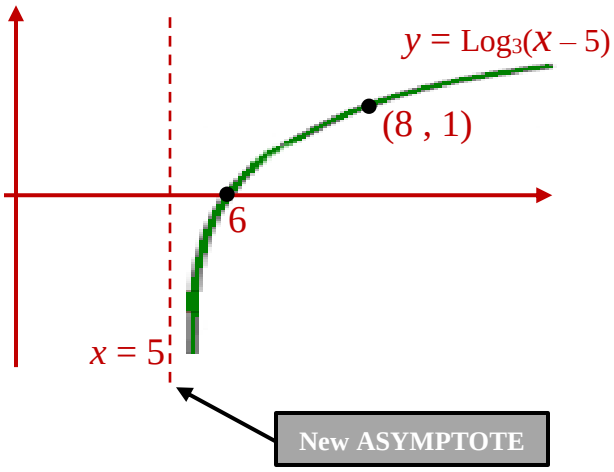
a) Shift the Graph **UP** 2



$(1 , 0) \rightarrow (1 , 2)$

$(3 , 1) \rightarrow (3 , 3)$

(b) Shift the Graph **RIGHT** 5



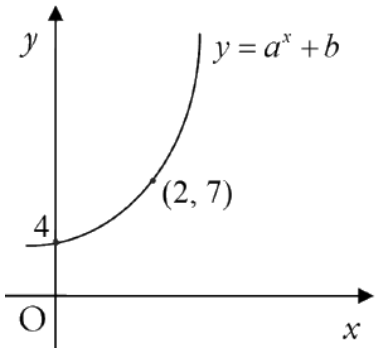
$(1 , 0) \rightarrow (6 , 0)$

$(3 , 1) \rightarrow (8 , 1)$

Examples - Finding Related Graph Equations:

- You may also be given the Graph and asked to find the equation of the curve.

10. For the graph opposite find the values of a and b and then state the curves equation.



Use (0 , 4): $4 = a^0 + b \Longrightarrow 4 = 1 + b \Longrightarrow b = 3$

Use (2 , 7): $7 = a^2 + 3 \Longrightarrow 4 = a^2 \Longrightarrow a = \sqrt{4}$

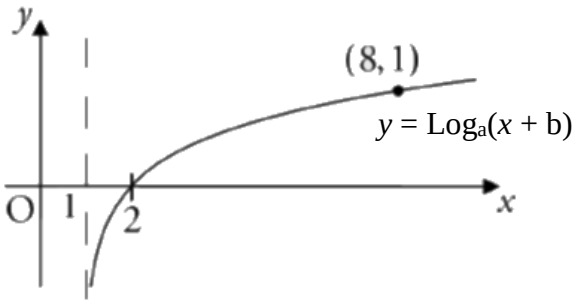
$\Longrightarrow a = \pm 2$

$\Longrightarrow a = 2$

So $y = 2^x + 3$

Since a > 0

11. For the graph opposite find the values of a and b and then state the curves equation.



Log curve usually passes through (1 , 0) but this passes through (0 , 2) so it has been shifted right by 1 $\Longrightarrow b = -1$

Use (8 , 1): $y = \text{Log}_a(x - 1)$

$\Longrightarrow 1 = \text{Log}_a(8 - 1)$

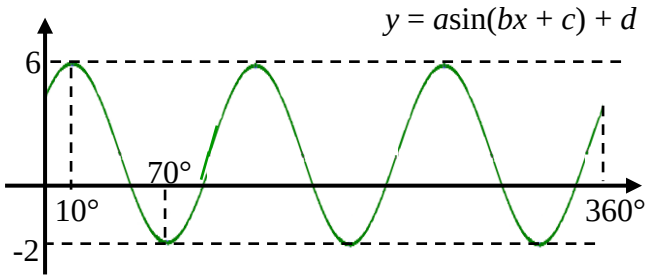
$\Longrightarrow 1 = \text{Log}_a 7$

$\Longrightarrow a = 7$

So $y = \text{Log}_7(x - 1)$

Since $\text{Log}_a a = 1$

12. For the graph opposite find the values of a , b, c and d and then state the curves equation.



a is half the height of the graph, so: $a = 4$

b is the number of cycles between 0 & 360, so: $b = 3$

d is how much it has been shifted vertically, so: $d = 2$

c is how much it has been shifted horizontally:

Sin graph is at its maximum at 90°

So sin3x is at its maximum at 30°

This curve is at its maximum at 10°

So it has been shifted 20° to the left, so $c = 20°$

So $y = 4\sin(3x + 20°) + 2$

Now attempt Exercise 8 from the Graphs of Functions booklet