Sets & Functions

SPTA Mathematics – Higher Notes

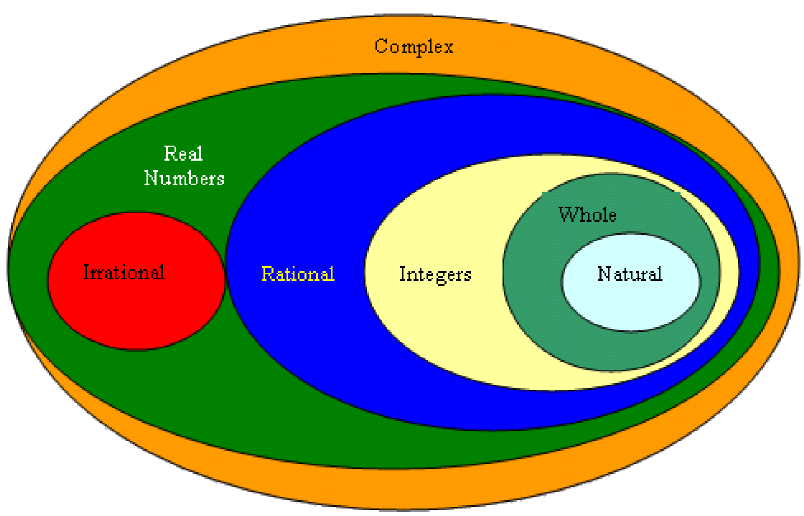
Sets:

- A **SET** refers to a group of elements (entries) that are linked in some way.
- These symbols mean; $\in$ "is a member of", $\notin$ "is NOT a member of" and $\subseteq$ "is a subset of"
- { } – curly brackets are used to define sets.
- There are 5 basic number sets used in Maths and 2 more advanced that you need to be aware of:

1. $\mathbb{N}$ – **NATURAL** numbers: {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, ……}
2. $\mathbb{W}$ – **WHOLE** numbers: {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, ……}
3. $\mathbb{Z}$ – **INTEGER** numbers: {….., –5, –4, –3, –2, –1, 0, 1, 2, 3, 4, 5, ……}
4. $\mathbb{Q}$ – **RATIONAL** numbers: Includes all integers, plus any number which can be written as a fraction.
5. $\mathbb{R}$ – **REAL** numbers: Includes all rational numbers, and irrational numbers
6. $\mathbb{I}$ – **IRATIONAL** numbers: Any number which cannot be written as a fraction such as $\sqrt{7}$ or $\boldsymbol{\pi}$
7. $\mathbb{C}$ – **COMPLEX** numbers: a **number** that can be expressed in the form $a + bi$, where a and b are real **numbers** and i is the imaginary unit, satisfying the equation $i^2 = -1$

   Complex numbers are NOT in the Higher.

- A VENN DIAGRAM can be used to show the link between all these sets:

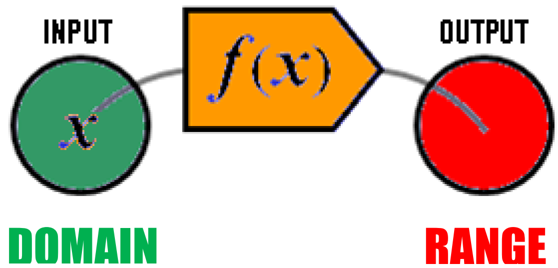


Functions:

- We have already looked at function calculations in National 5.
- A **FUNCTION** is a mathematical rule which takes a value in, carries out a process on that value and gives out a result.
- **FUNCTION NOTATION**: $f(x)$ read as "f of x" where x is the number going into the function.
- Functions can be called anything, with the most common being; $f(x)$, $g(x)$, $h(x)$, $k(x)$, etc

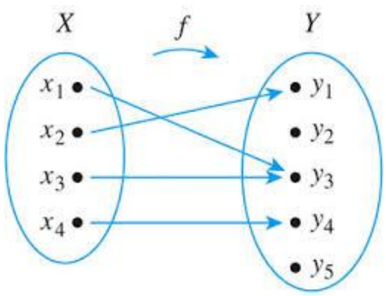
Now attempt Exercise 1 from the Functions booklet

- A diagram can be used to explain what a Function is, as shown below:



- In Maths the INPUT is called the **DOMAIN** and the OUTPUT is known as the **RANGE**.
- Every element in the DOMAIN must match with **EXACTLY ONE** element in the RANGE.

  The arrow diagram below show what a function looks like:



Set X is the DOMAIN

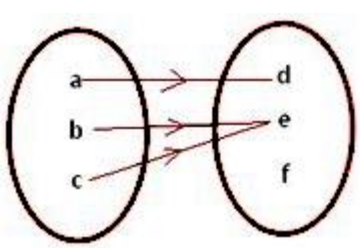
Set Y is the RANGE.

This is a Function as every element in the Domain is mapped to exactly one element in the Range.

Examples:

1. State whether these diagrams represent functions or not and give a reason:

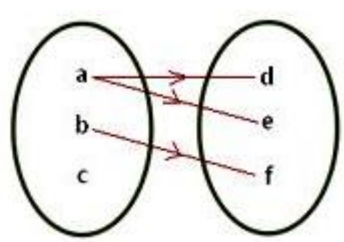
a)



FUNCTION

Every element in the Domain maps to exactly one element in the Range. The fact that elements "b & c" map to the same element is not an issue!!

(b)

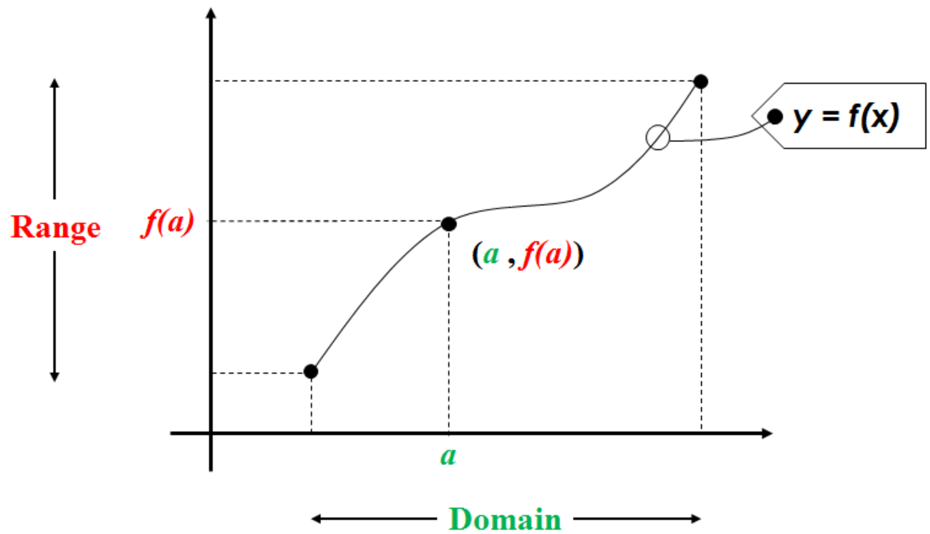


NOT A FUNCTION

Element "c" in the Domain doesn't map to any element in the Range and element "a" maps to 2 elements hence not a function!

Graphs of Functions:

- Graphs of a function can be drawn by making $y = f(x)$.
- The x-axis is the Domain and the y-axis is the Range.
- To determine whether or not a graph represents a function we can use the vertical line test:
  - Draw a Vertical Line
  - If it intersects with the Graph at one point only it's a Function
  - If it intersects at more than one point its NOT a Function.
- A Graph of a Function is shown below:

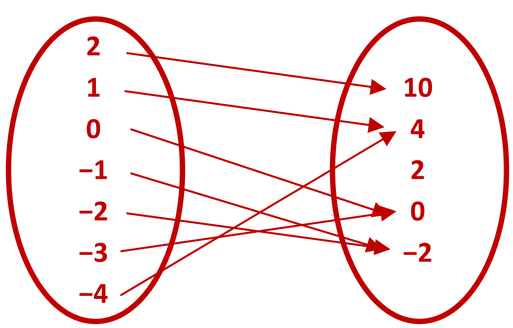


Examples:

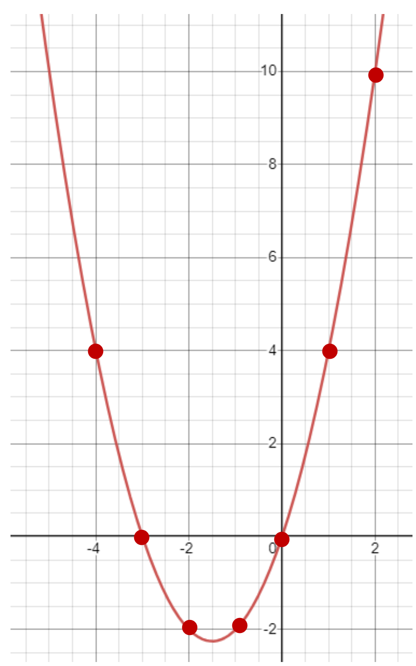
2. $g(x) = x^2 + 3x$ where $x \in \{-4, -3, -2, -1, 0, 1, 2\}$

a) State the Range. $g(x) \in \{-2, 0, 4, 10\}$

b) Draw an arrow diagram

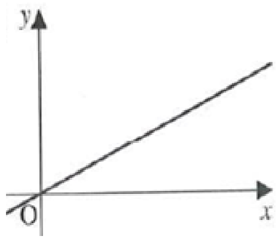


c) Draw a Graph of the function.



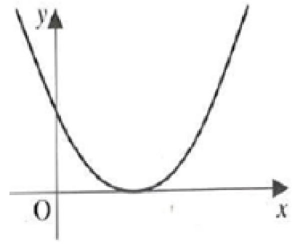
3. Which of the following graphs below represent functions?

a)



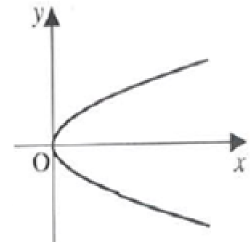
FUNCTION

(b)



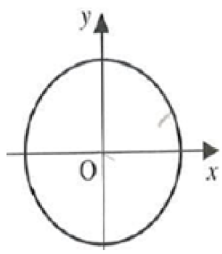
FUNCTION

(c)



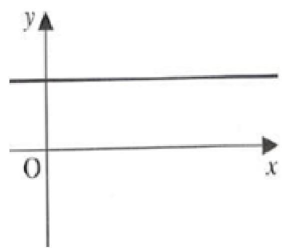
NOT a FUNCTION

d)



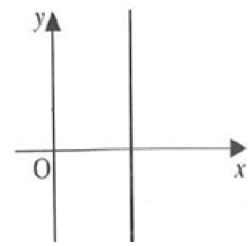
NOT a FUNCTION

(e)



FUNCTION

(f)



NOT a FUNCTION

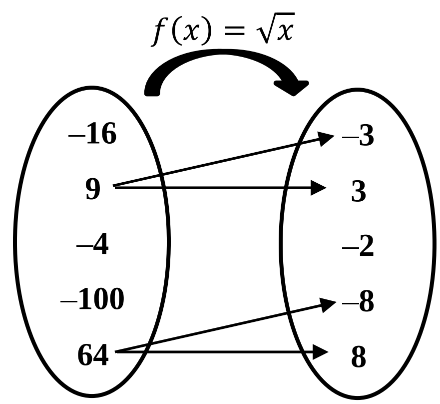
Restrictions:

- Is $f(x) = \sqrt{x}$ a function?

  Lets look at its ARROW diagram:

  Without any restrictions the DOMAIN is $x \in \mathbb{R}$
  i.e. x can take any real number as its value.

  As we can see from this small sample $f(x) = \sqrt{x}$ is **NOT a FUNCTION** as negatives cannot be square rooted and each positive number has 2 solutions!!!!



- We can make $f(x) = \sqrt{x}$ a function by restricting the size of the Domain and the Range.
  We need to restrict the Domain to the positive numbers: $\{x \in \mathbb{R} : x \geq 0\}$
  We also need to restrict the Range to the positive numbers: $f(x) \geq 0$

- In maths it is quite normal to restrict the Domain to get a function, the Range is less common.

- Some common Domain Restrictions are:
  - You cannot square root a negative as mentioned above.
  - In a fraction you cannot divide by zero, e.g. $f(x) = \frac{5x^2-3}{2x-8}$, $\{x \in \mathbb{R} : x \neq 4\}$
  - You cannot take a log of zero or a negative, e.g. $f(x) = \log_a x$, $\{x \in \mathbb{R} : x > 0\}$
  - In trig Tan(90) and Tan(270) don't exist, e.g. $f(x) = \tan(x + 30)$, $\{x \in \mathbb{R} : x \neq 60, 240\}$

Now attempt Exercise 2 from the Functions booklet

Examples:

4. Describe a suitable Domain for these functions:

a) $f(x) = \sqrt{x-5}$

$x - 5 \geq 0$

$x \geq 5$

$\{x \in \mathbb{R} : x \geq 5\}$

Cannot sqaure root a negative

(b) $f(x) = \frac{x+1}{x^2+x-6}$

Cannot divide by zero

$x^2 + x - 6 \neq 0$

$(x+3)(x-2) \neq 0$

$x + 3 \neq 0 \quad x - 2 \neq 0$

$x \neq -3 \quad x \neq 2$

$\{x \in \mathbb{R} : x \neq -3, 2\}$

Your being asked in Question 4 above for what values does the function work. However Question 4 could be asked slightly differently to find for what values does the function NOT work as shown below:

4. For what values of x are these functions undefined?

a) $f(x) = \sqrt{x-5}$

$x - 5 < 0$

$x < 5$

Cannot sqaure root a negative

(b) $f(x) = \frac{x+1}{x^2+x-6}$

Cannot divide by zero

$x^2 + x - 6 = 0$

$(x+3)(x-2) = 0$

$x + 3 = 0 \quad x - 2 = 0$

$x = -3 \quad x = 2$

As stated earlier it is not as common to restrict the RANGE as it is the DOMAIN, however you could be asked to write down the Range for a given function as shown in this example below:

5. Write down a suitable Range for the following functions:

a) $f(x) = x^2$

When a number is squared the answer is always positive or zero so: $f(x) \geq 0$

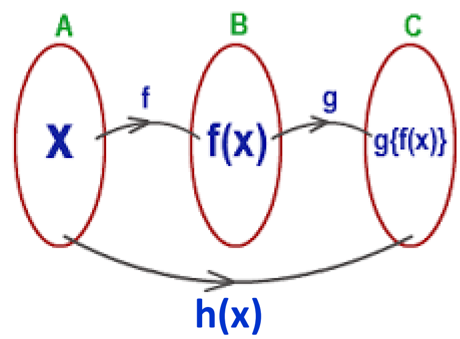
b) $f(x) = 2\cos x$

From National 5 we know that the cos curve goes between –1 & 1 so: $-2 \leq f(x) \leq 2$

Now attempt Exercise 3 from the Functions booklet

Composite Functions:

- 2 or more Functions can be combined to form a new function.
- These new functions are called **COMPOSITE FUNCTIONS**.
- If we have 2 functions, $f(x)$ and $g(x)$ then $h(x) = f(g(x))$ & $k(x) = g(f(x))$ are composite functions.
- We read $f(g(x))$ as "f of g of x" and $g(f(x))$ as "g of f of x"
- For **most** functions $f(g(x))$ wil **NOT** be the same as $g(f(x))$
- Composite functions can be illustrated by an arrow diagram as shown here:
  - $h(x) = f(g(x))$
  - A is the DOMAIN for $f(x)$ and $h(x)$
  - B is the RANGE for $f(x)$ and the DOMAIN for $g(x)$
  - C is the RANGE for $g(x)$ and $h(x)$



Examples:

6. If $f(x) = 2x$ and $g(x) = x + 15$

a) Evaluate $f(g(3))$

$g(3) = 3 + 15$
$= 18$
$f(g(3)) = f(18)$
$= 2 \times 18$
$= 36$

(b) Find a formula for $h(x) = f(g(x))$

$h(x) = f(g(x))$
$= f(x + 15)$
$= 2(x + 15)$
$= 2x + 30$

7. If $f(x) = x^2$ and $g(x) = x + 3$

a) Evaluate $f(g(5))$

$g(5) = 5 + 3$
$= 8$
$f(g(5)) = f(8)$
$= 8^2$
$= 64$

(b) Find a formula for $f(g(x))$

$f(g(x)) = f(x + 3)$
$= (x + 3)^2$
$= x^2 + 6x + 9$

Not the same!!!

c) Evaluate $g(f(5))$

$f(5) = 5^2$
$= 25$
$g(f(5)) = g(25)$
$= 25 + 3$
$= 28$

(d) Find a formula for $g(f(x))$

$g(f(x)) = g(x^2)$
$= x^2 + 3$

Now attempt Exercise 4A from the Functions booklet

8 . If $f(x) = x^2 + 3$ and $g(x) = 2x - 5$, Find a formula for:

a) $f(g(x))$

$$f(g(x)) = f(2x-5)$$
$$= (2x-5)^2 + 3$$
$$= 4x^2 - 20x + 25 + 3$$
$$= 4x^2 - 20x + 28$$

(b) $f(f(x))$

$$f(f(x)) = f(x^2+3)$$
$$= (x^2+3)^2 + 3$$
$$= x^4 + 6x^2 + 9 + 3$$
$$= x^4 + 6x^2 + 12$$

c) $g(f(x))$

$$g(f(x)) = g(x^2+3)$$
$$= 2(x^2+3) - 5$$
$$= 2x^2 + 6 - 5$$
$$= 2x^2 + 1$$

(d) $g(g(x))$

$$g(g(x)) = g(2x-5)$$
$$= 2(2x-5) - 5$$
$$= 4x - 10 - 5$$
$$= 4x - 15$$

9 . If $f(x) = \frac{7}{2x^2-8}$ and $g(x) = x + 3$, Find a formula for:

a) $f(g(x))$

$$f(g(x)) = f(x+3)$$
$$= \frac{7}{2(x+3)^2-8}$$
$$= \frac{7}{2(x^2+6x+9)-8}$$
$$= \frac{7}{2x^2+12x+18-8}$$
$$= \frac{7}{2x^2+12x+10}$$

(b) State a suitable DOMAIN for $f(g(x))$

$$2x^2 + 12x + 10 \neq 0$$
$$x^2 + 6x + 5 \neq 0$$
$$(x+5)(x+1) \neq 0$$
$$x + 5 \neq 0 \text{ or } x + 1 \neq 0$$
$$x \neq -5 \text{ or } x \neq -1$$
$$\{x \in \mathbb{R} : x \neq -5, -1\}$$

10 . If $f(x) = 2x + 5$ and $g(x) = cosx$, Find a formula for:

a) $f(g(x))$

$$f(g(x)) = f(cosx)$$
$$= 2(cosx) + 5$$

(b) $g(f(x))$

$$g(f(x)) = g(2x+5)$$
$$= \cos(2x+5)$$

11 . If $(x) = \frac{x}{x-1}$, Find a formula for $f(f(x))$

$$f(f(x)) = f\left(\frac{x}{x-1}\right)$$

$$= \frac{\left(\frac{x}{x-1}\right)}{\left(\frac{x}{x-1}\right)-1}$$

$$\frac{x}{x-1} - 1 = \frac{x}{x-1} - \frac{1}{1}$$

$$= \frac{x}{x-1} - \frac{x-1}{x-1}$$

$$= \frac{x-(x-1)}{x-1}$$

$$= \frac{1}{x-1}$$

$$= \frac{\frac{x}{x-1}}{\frac{1}{x-1}}$$

$$= \frac{x}{x-1} \div \frac{1}{x-1}$$

$$= \frac{x}{x-1} \times \frac{x-1}{1}$$

$$= \frac{x(x-1)}{x-1}$$

$$= x$$

Now attempt Exercise 4B from the Functions booklet

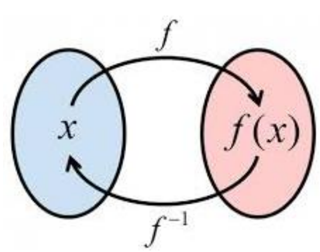
Inverse Functions:

- An **INVERSE FUNCTION** is the opposite or reverse of the original function.

- A function has an inverse function if it is a **one to one correspondence**, i.e. each term in the domain matches to exactly one term in the range and vice versa.

- The inverse function of $f(x)$ is denoted as $f^{-1}(x)$.

- If a number, x, is put into a function $f(x)$ followed by $f^{-1}(x)$ then the result will just be x again. This can be shown in a diagram as follows:



- Finding the inverse of a function is the same process as changing the subject from National 5.

- The Graph of an inverse function is a reflection of the original in the line $y = x$.

Examples:

12. If $(x) = 2x + 4$, Find the inverse function $f^{-1}(x)$

$$f(x) = 2x + 4$$

$$y = 2x + 4$$ ← **Replace $f\,(x)$ with y**

$$y - 4 = 2x$$

$$\frac{y-4}{2} = x$$ ← **Change the subject to x**

$$f^{-1}(x) = \frac{x-4}{2}$$ ← **Replace x with $f^{-1}(x)$ and y with x**

You can check your answer by subbing in any number to see if you return to the start again as folllows:

$$f(3) = 2(3) + 4$$

$$f(3) = 6 + 4$$

$$f(3) = 10$$

$$f^{-1}(10) = \frac{10-4}{2}$$

$$f^{-1}(10) = \frac{6}{2}$$

$$f^{-1}(10) = 3$$ ← **What we started with so inverse is correct**

13. If $f(x) = \frac{x^2-5}{4}$, Find the inverse function $f^{-1}(x)$

$$f(x) = \frac{x^2-5}{4}$$

$$y = \frac{x^2-5}{4}$$

$$4y = x^2 - 5$$

$$4y + 5 = x^2$$

$$\sqrt{4y+5} = x$$

$$f^{-1}(x) = \sqrt{4x+5}$$

14 . If $f(x) = \frac{1}{x+1}$ and $g(x) = \frac{1}{x} - 1$

a) Find a formula for $f(g(x))$

$f(g(x)) = f\left(\frac{1}{x} - 1\right)$

$f(g(x)) = f\left(\frac{1}{x} - 1\right)$

$f(g(x)) = \frac{1}{\left(\frac{1}{x} - 1\right) + 1}$

$f(g(x)) = \frac{1}{\frac{1}{x}}$

$f(g(x)) = 1 \div \frac{1}{x}$

$f(g(x)) = 1 \times \frac{x}{1}$

$f(g(x)) = x$

(b) State the relationship between the 2 functions

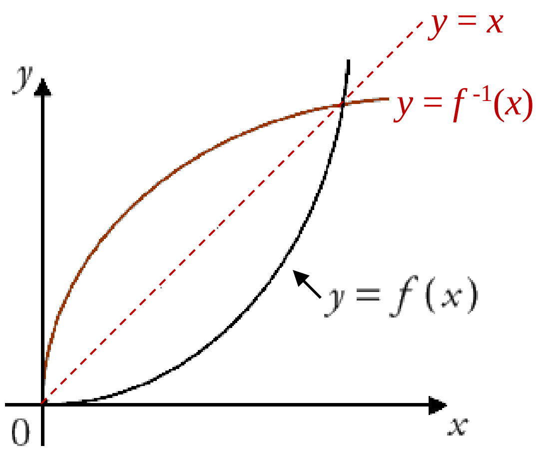
Since $f(g(x)) = x$ the 2 functions are inverses

i.e. $g(x) = f^{-1}(x)$

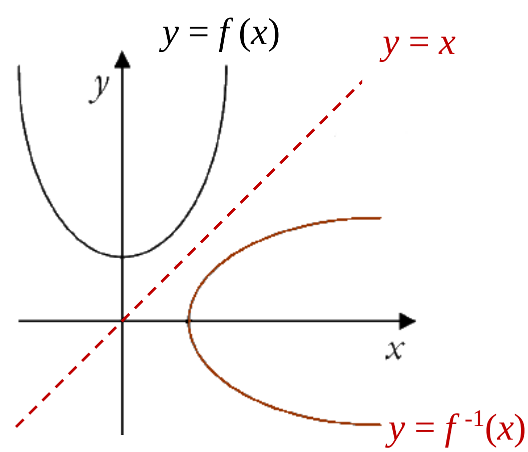
Now attempt Exercises 5 & 6A/B from the Functions booklet

15 . Sketch the inverse graph of:

a)



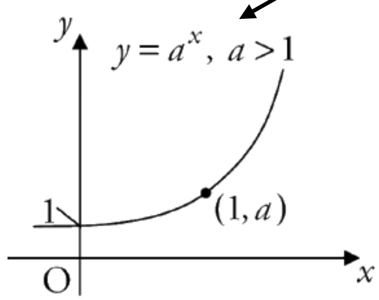
(b)

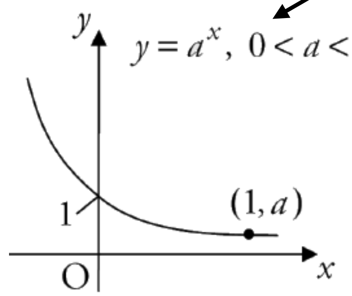


Now attempt Exercise 7 from the Functions booklet

Exponential Functions:

- An exponential equation (function) to the base a is in the form: $y = a^x$ (or $f(x) = a^x$), $a > 0$
- When $x = 0$ then $y = a^0 = 1$ and when $x = 1$ then $y = a^1 = a$ hence the graph of an exponential will always pass through the 2 points: (0 , 1) and (1 , a). ← **Must learn**
- If $a > 1$ then the graph is called a GROWTH curve and when $0 < a < 1$ it is called a DECAY curve:



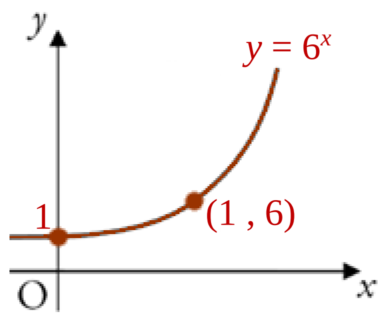


- The graph gets very close to the x-axis, but never touches it. The x-axis is said to be an ASYMPTOTE to the curve.
- If we know 2 points on the graph we can find the equation of the curve.
- $f(x) = e^x$ is called the exponential function to the base e. It is sometimes written as $exp(x)$.
- Your calculator will have an e^x button, usually above the ln button(see logarithms below).
- The constant $e = 2.718281828...$ is an important number in Maths and like π it is irrational.

Examples:

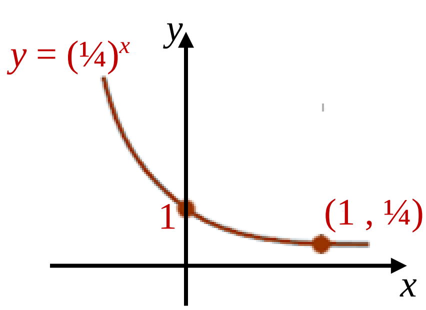
16. Sketch the curve $y = 6^x$

Since $a = 6$ the graph passes through the points: (0 , 1) and (1 , 6)



17. Sketch the curve $y = (¼)^x$

Since $a = ¼$ the graph passes through the points: (0 , 1) and (1 , ¼)

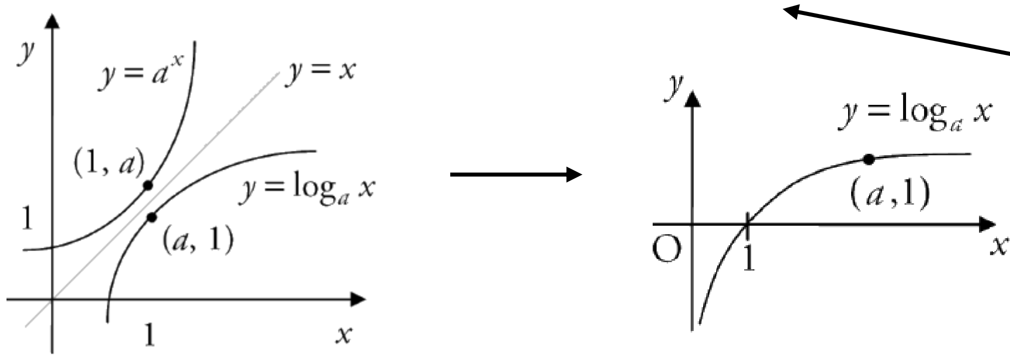


Now attempt Exercise 8 from the Functions booklet

Logarithmic Functions:

- A Logarithmic equation(function), shortened to *Log*, is the inverse of an exponential function.
- A *Log* equation (function) to the base a is written in the form: $y = Log_a x$ (or $f(x) = Log_a x$), $a > 0$.
- As it is the inverse of exponentials its graph is the refelection of a GROWTH curve in the line $y = x$.
- Hence a *Log* curve will always pass through the points: (1 , 0) and (a , 1) as shown below:

Must learn

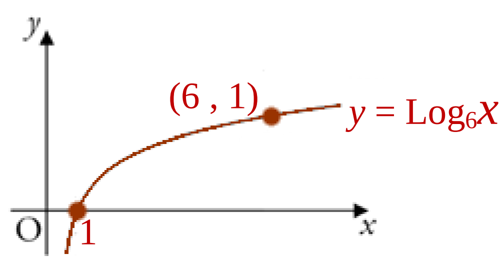


- The graph gets very close to the y-axis, but never touches it. The y-axis is said to be an ASYMPTOTE to the curve.
- In general: $Log_a 1 = 0$ (*Log* to any base of 1 is zero) and
  $Log_a\, a = 1$ (*Log* to base a of a is one).
- If we know 2 points on the graph we can find the equation of the curve.
- The *Log* key on your calculator is to the base 10, i.e. Log_{10}
- Log_e , usually written as *ln*, is called the NATURAL LOGARITHM and is also on the calculator.

Examples:

18. Sketch the curve $y = Log_6 x$

Since $a = 6$ the graph passes through the points: (1 , 0) and (6 , 1)



Now attempt Exercises 9 & 10 from the Functions booklet