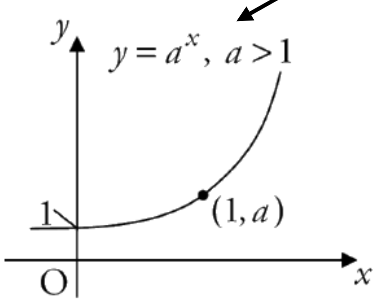
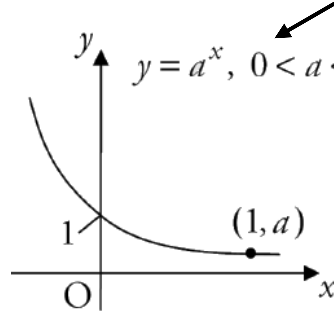
H Logs & Exp.

SPTA Mathematics – Higher Notes

Exponential – We have seen most of this already:

- An exponential equation (function) to the base a is in the form: $y = a^x$ (or $f(x) = a^x$), $a > 0$
- When $x = 0$ then $y = a^0 = 1$ and when $x = 1$ then $y = a^1 = a$ hence the graph of an exponential will always pass through the 2 points: $(0, 1)$ and $(1, a)$. ← **Must learn**
- If $a > 1$ then the graph is called a GROWTH curve and when $0 < a < 1$ it is called a DECAY curve:



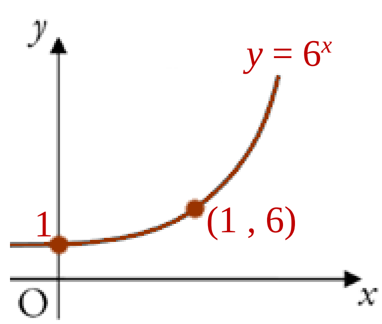


- The graph gets very close to the x-axis, but never touches it. The x-axis is said to be an ASYMPTOTE to the curve.
- If we know 2 points on the graph we can find the equation of the curve.
- $f(x) = e^x$ is called the exponential function to the base e. It is sometimes written as $exp(x)$.
- Your calculator will have an e^x button, usually above the *ln* button(see logarithms below).
- The constant $e = 2.718281828...$ is an important number in Maths and like π it is irrational.

Examples:

1. Sketch the curve $y = 6^x$

Since $a = 6$ the graph passes through the points: $(0, 1)$ and $(1, 6)$



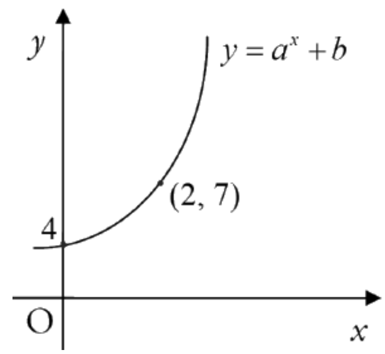
2. For the graph below find the values of a and b and then state the curves equation.

Use (0 , 4): $4 = a^0 + b \Longrightarrow 4 = 1 + b \Longrightarrow b = 3$

Use (2 , 7): $7 = a^2 + 3 \Longrightarrow 4 = a^2 \Longrightarrow a = \sqrt{4}$

$\Longrightarrow a = 2$

So $y = 2^x + 3$



3. Use your calculator to find $f(x) = e^x$, when $x = 5$:

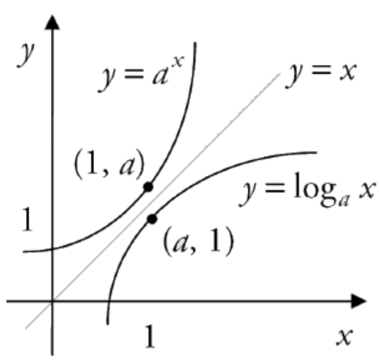
$f(5) = e^5 = 148.41$

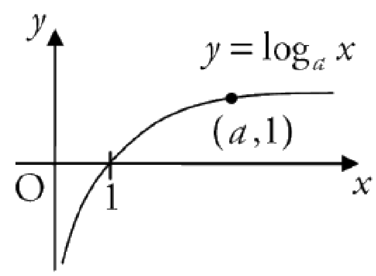
Now attempt Exercise 1 from the Logs & Exponentials booklet.

Logarithms – We have seen most of this already:

- A Logarithmic equation (function), shortened to *Log*, is the inverse of an exponential function.
- A *Log* equation (function) to the base a is written in the form: $y = Log_a x$ (or $f(x) = Log_a x$), $a > 0$.
- As it is the inverse of exponentials its graph is the reflection of a GROWTH curve in the line $y = x$.
- Hence a *Log* curve will always pass through the points: (1 , 0) and (a , 1) as shown below:

Must learn





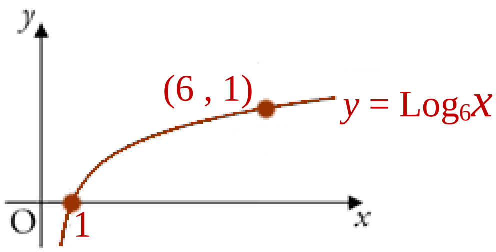
- The graph gets very close to the y-axis, but never touches it. The y-axis is said to be an ASYMPTOTE to the curve.
- In general: $Log_a 1 = 0$ (*Log* to any base of 1 is zero) and
  $Log_a a = 1$ (*Log* to base a of a is one).
- If we know 2 points on the graph we can find the equation of the curve.
- The *Log* key on your calculator is to the base 10, i.e. Log_{10}
- Log_e , usually written as *ln*, is called the **NATURAL LOGARITHM** and is also on the calculator.

LN not IN!!

Examples:

4. **a)** Sketch the curve $y = Log_6 x$

Since $a = 6$ the graph passes through the points: (1 , 0) and (6 , 1)

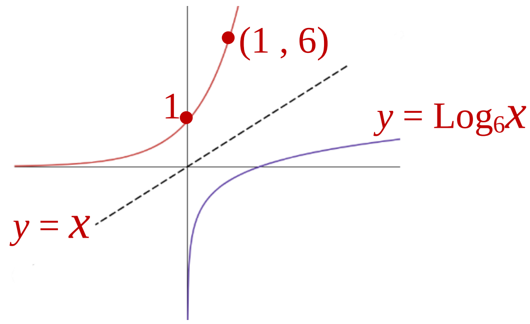


(b) Sketch the inverse of $y = Log_6 x$

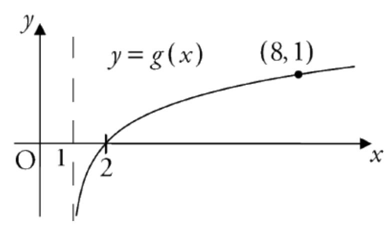
Reflect curve in the line $y = x$

So (1 , 0) $\rightarrow$ (0 , 1)

(6 , 1) $\rightarrow$ (1 , 6)



5. A graph of the function $g(x) = Log_a(x + b)$, where a, b are constants and $a > 1$ is shown here:



Find the values of a and b and state $g(x)$.

Curve normally passes through (1 , 0), but is passing through (2 , 0), so has been shifted right 1, giving $b = -1$.

Use (8 , 1): $1 = Log_a(8 - 1) \implies Log_a 7 = 1 \implies a = 7$

So $g(x) = Log_7 x$

6. Use your calculator to find:

a) $Log_{10}75 = 1.875$

(b) $ln75 = 4.3175$

Now attempt Exercise 2 from the Logs & Exponentials booklet.

Exponentials ↔ Logarithms:

- $y = Log_a x \leftrightarrow x = a^y$, where a , $x > 0$ ← **Must learn**

Examples:

7. Express each of the following in exponential form and solve if possible:

a) $Log_5\, 125 = 3 \implies 125 = 5^3$

b) $Log_3\, x = y \implies x = 3^y$

c) $Log_2\, x = 3 \implies x = 2^3 \implies x = 8$

d) $Log_3\, (2x - 5) = 2$

$$\begin{aligned} &\implies 2x - 5 = 3^2 \\ &\implies 2x - 5 = 9 \\ &\implies 2x = 14 \\ &\implies x = 7 \end{aligned}$$

8. Express each of the following in Logarithmic form:

a) $2^3 = 8 \implies Log_2\, 8 = 3$

b) $729 = 3^6 \implies Log_3\, 729 = 6$

c) $x = 8^y \implies Log_8\, x = y$

Now attempt Exercise 3 from the Logs & Exponentials booklet.

Laws of Logarithms:

- Remember that $Log_a\ a = 1$ and $Log_a\ 1 = 0$ **Must learn**
- There are 3 Laws of Logarithms that you must remember, they are **NOT** on the Formulae Sheet:
  1. $Log_a\, x + Log_a\, y = Log_a\, xy$, where $a , x , y > 0$
  2. $Log_a\, x - Log_a\, y = Log_a\, {}^{x}/_{y}$, where $a , x , y > 0$ **Must learn**
  3. $Log_a\, x^n = nLog_a\, x$, where $a , x > 0$
- For Laws 1 & 2 the bases of each *Log* must be same.
- We can use the Laws of Logarithms to solve equations involving Logs.
- When there are several additions or subtractions of Logs they can be combined into one as follows:

$$\log_a\left(\frac{\text{arguments of } \textbf{positive} \text{ log terms}}{\text{arguments of } \textbf{negative} \text{ log terms}}\right)$$

Examples:

9. Solve the following:

a) $Log_3\, 1 + Log_3\, 9$

$= Log_3\,(1 \times 9)$ **Rule 1**

$= Log_3\, 9$

$= Log_3\, 3^2$

$= 2\, Log_3\, 3$ **Rule 3**

$= 2 \times 1$

$= 2$

(b) $Log_4\, 6 - Log_4\, 3$

$= Log_4\, {}^{6}/_{3}$ **Rule 2**

$= Log_4\, 2$

$= Log_4\, 4^{1/2}$

$= {}^{1}/_{2}\, Log_4\, 4$ **Rule 3**

$= {}^{1}/_{2} \times 1$

$= {}^{1}/_{2}$

c) $Log_6\, 4 + 2\, Log_6\, 3$

$= Log_6\, 4 + Log_6\, 3^2$ **Rule 3**

$= Log_6\, 4 + Log_6\, 9$

$= Log_6\,(4 \times 9)$ **Rule 1**

$= Log_6\, 36$

$= Log_6\, 6^2$

$= 2\, Log_6\, 6$ **Rule 3**

$= 2 \times 1$

$= 2$

(d) $Log_2\, 3 - Log_2\, 6 + Log_2\, 2 - Log_2\, 8$

$= Log_2\,({}^{3 \times 2}/_{6 \times 8})$ **Rule 1 & 2**

$= Log_2\,({}^{6}/_{48})$

$= Log_2\,({}^{1}/_{8})$

$= Log_2\,({}^{1}/_{2^3})$

$= Log_2\, 2^{-3}$

$= -3\, Log_2\, 2$ **Rule 3**

$= -3 \times 1$

$= -3$

10. Simplify $4Log_e(2e) - 3Log_e(3e)$, expressing your answer in the form $a + Log_e\, b - Log_e\, c$, where a , b and c are whole numbers.

$4Log_e\,(2e) - 3Log_e\,(3e)$

$= Log_e\,(2e)^4 - Log_e\,(3e)^3$

$= Log_e\,(16e^4) - Log_e\,(27e^3)$

$= Log_e\left(\frac{16e^4}{27e^3}\right)$

$= Log_e\left(\frac{16e}{27}\right)$

$= Log_e\,(16e) - Log_e\,27$

$= Log_e\,16 + Log_e\,e - Log_e\,27$

$= 1 + Log_e\,16 - Log_e\,27$

OR

$4Log_e\,(2e) - 3Log_e\,(3e)$

$= 4(Log_e\,2 + Log_e\,e) - 3(Log_e\,3 + Log_e\,e)$

$= 4Log_e\,2 + 4Log_e\,e - 3Log_e\,3 - 3Log_e\,e$

$= Log_e\,2^4 + Log_e\,e - Log_e\,3^3$

$= Log_e\,16 + 1 - Log_e\,27$

$= 1 + Log_e\,16 - Log_e\,27$

Now attempt Exercise 4A from the Logs & Exponentials booklet.

11. Solve:

a) $Log_a\, 13 + Log_a\, x = Log_a\, 273$

$Log_a\, 13x = Log_a\, 273$

$13x = 273$

$x = 21$

Cancel Logs on both sides.

(b) $Log_2\, 7 = Log_2\, x + 3$

$Log_2\, 7 - Log_2\, x = 3$

$Log_2\left(\frac{7}{x}\right) = 3$

$^7/_x = 2^3$

$7 = 8x$

$x = {}^7/_8$

Convert Logs to exponential.

c) $Log_{11}(4x + 3) - Log_{11}(2x - 3) = 1$

$Log_{11}\left(\frac{4x+3}{2x-3}\right) = 1$

$\frac{4x+3}{2x-3} = 11^1$

$4x + 3 = 11(2x - 3)$

$4x + 3 = 22x - 33$

$-18x + 3 = -33$

$-18x = -36$

$x = 2$

(d) $e^x = 7$

$\ln e^x = \ln 7$

$x \ln e = \ln 7$

$x = \ln 7$

$x = 1.9$

Take Logs of both sides.

12. Solve $Log_a(2p + 1) + Log_a(3p - 10) = Log_a(11p)$ for $p > 4$

$Log_a(2p + 1) + Log_a(3p - 10) = Log_a(11p)$

$Log_a[(2p + 1)(3p - 10)] = Log_a(11p)$

$(2p + 1)(3p - 10) = 11p$

$6p^2 - 17p - 10 = 11p$

$6p^2 - 17p - 10 - 11p = 0$

$6p^2 - 28p - 10 = 0$

$(3p + 1)(p - 5) = 0$

$3p + 1 = 0$ or $p - 5 = 0$

$p = -{}^1/_3, 5$ so $p = 5$ since $p > 4$

Cancel Logs on both sides.

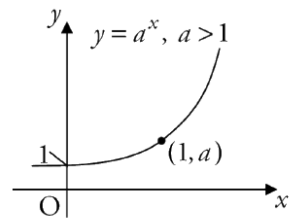
Now attempt Exercise 4B from the Logs & Exponentials booklet.

Growth & Decay:

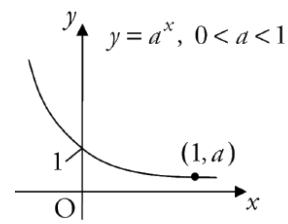
- Exponential growth or decay are processes that take place over time, in which the rate of change of the function, i.e. the derivative of the function, is proportional to the current value.

- These functions appear in many different spheres, including the natural world, science, computing and finance.

- Examples of exponential growth could be the increase of a financial investment, the spread of a virus, the sales of a new toy or gadget or the ‘going viral’ of a video, a post, a news article or a meme on the internet.



- Examples of exponential decay include many types of chemical reactions and the decay of radioactive materials.



- When working with exponential growth and decay, we will primarily use the exponential function ($\boldsymbol{e^x}$ or **exp** $\boldsymbol{x}$) and the natural log function (**ln** $\boldsymbol{x}$ or **log**$\boldsymbol{e}$ $\boldsymbol{x}$).

Examples:

13. For the formula the speed, s, of an object at time, t minutes is given by: $s(t) = 100e^{3t}$

a) Evaluate the initial speed.

$$s(0) = 100e^{3(0)}$$
$$= 100e^{0}$$
$$= 100$$

(b) Calculate the time taken for the initial speed to be trebled.

$$s(t) = 100e^{3t}$$
$$300 = 100e^{3t}$$
$$e^{3t} = 3$$
$$ln\ e^{3t} = ln\ 3$$

Take Logs of both sides.

$$3t\ ln\ e = 1.099$$
$$3t = 1.099$$
$$t = 0.366 \text{ minutes}$$

14. A radioactive substance has Mass, M grams, at t years such that M is given by the formula:

$M_t = M_0 e^{-0.03t}$ where M_0 is the initial mass in grams.

a) If M_0 = 800g find the mass after 7 years.

$$M_7 = 800e^{-0.03 \times 7}$$
$$= 800e^{-0.21}$$
$$= 648.47\text{g}$$

(b) Calculate the half-life of this substance.

$$400 = 800e^{-0.03t}$$
$$0.5 = e^{-0.03t}$$
$$\ln e^{-0.03t} = \ln 0.5 \leftarrow \text{Take Logs of both sides.}$$
$$-0.03t \ln e = -0.69315$$
$$-0.03t = -0.69315$$
$$t = 23.105 \text{ years}$$

Sometimes you are not given the initial value within the question, but still be asked to find the half-life!!

Cancel M_o and continue as above. →

$$\tfrac{1}{2} M_o = M_o e^{-0.03t}$$
$$0.5 = e^{-0.03t}$$

15. The amount of radioactive substance in a sample of contaminated soil after t years is given by the equation $A(t) = A_0e^{-0.0025t}$ where A_0 is the initial amount. ← **Not told the initial amount!!**

a) Calculate how long it will take for the substance to reach 60% of the initial value.

$0.6A_o$

$$A(t) = A_0e^{-0.0025t}$$
$$0.6A_0 = A_0e^{-0.0025t}$$
$$0.6 = e^{-0.0025t}$$
$$\ln e^{-0.0025t} = \ln 0.6$$
$$-0.0025t \ln e = -0.51082$$
$$-0.0025t = -0.51082$$
$$t = 204.33 \text{ years}$$

b) After 40 years there was 78 grams of radiation in the sample. Calculate the amount of substance in the soil during initial observations.

$$(t) = A_0e^{-0.0025t}$$
$$78 = A_0e^{-0.0025(40)}$$
$$78 = A_0e^{-0.1}$$
$$A_0 = \frac{78}{e^{-0.1}}$$
$$A_0 = 86.2 \text{ grams}$$

Now attempt Exercise 5 from the Logs & Exponentials booklet.

Experimental Data:

- Often Science experiments result in graphs of the forms: $y = kx^n$ or $y = ab^x$
- It is often more useful to have this information in the form of a straight line as it is then easier to perform calculations on. We can transform these equations into the form $Y = mX + C$ using Logs.
- Type 1: $y = kx^n$

Use the Base of Logs shown on graph.

$$\begin{aligned} y &= kx^n \\ Log\, y &= Log(kx^n) \\ Log\, y &= Log\, k + Log\, x^n \\ Log\, y &= Log\, k + n\, Log\, x \\ Log\, y &= n\, Log\, x + Log\, k \\ Y &= m\, X + C \end{aligned}$$

Type 2: $y = ab^x$

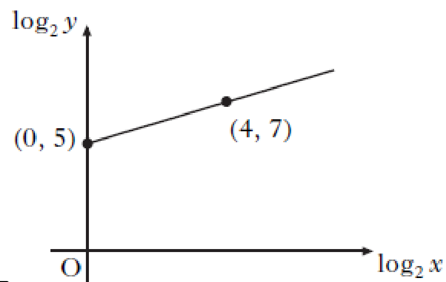
$$\begin{aligned} y &= ab^x \\ Log\, y &= Log(ab^x) \\ Log\, y &= Log\, a + Log\, b^x \\ Log\, y &= Log\, a + x\, Log\, b \\ Log\, y &= (Log\, b)\, x + Log\, a \\ Y &= \mathrm{M}\, x + C \end{aligned}$$

Examples:

16. Type 1: Variables x and y are related by the equation $y = kx^n$

The graph of $Log_2\, y$ against $Log_2\, x$ is a straight line through (0 , 5) and (4 , 7) as shown in the diagram.

Find the values of k and n and state the equation.



$$\begin{aligned} y &= kx^n \\ Log_2\, y &= Log_2\,(kx^n) \\ Log_2\, y &= Log_2\, k + Log_2\, x^n \\ Log_2\, y &= Log_2\, k + n\, Log_2\, x \\ Log_2\, y &= n\, Log_2\, x + Log_2\, k \end{aligned}$$

So $Y = mX + C$

Use coordinates to find Gradient.

$$m = \frac{y_B - y_A}{x_B - x_A}$$

$$n = \frac{7-5}{4-0}$$

$$= \frac{2}{4} = \frac{1}{2}$$

$$n = \frac{1}{2}$$

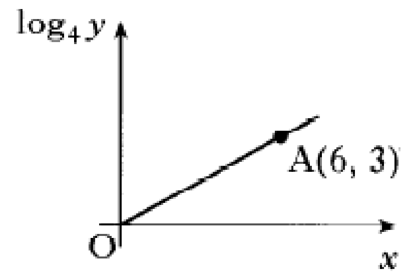
y-intercept

$$\begin{aligned} C &= 5 \\ Log_2\, k &= 5 \\ k &= 2^5 \\ k &= 32 \end{aligned}$$

So $y = 32x^{1/2}$

17. Type 2: Two variables x and y are connected by the law $y = ab^x$. The graph of $Log_4\, y$ against x is a straight line through the origin and (6 , 3).

Find the values of a and b and state the equation.



$$y = ab^x$$

$$Log_4\, y = Log_4\, a + Log_4\, b^x$$

$$Log_4\, y = Log_4\, a + x\, Log_4\, b$$

$$Log_4\, y = (Log_4\, b)\, x + Log_4\, a$$

$$Y = M\, x + C$$

$$m = \frac{y_B - y_A}{x_B - x_A}$$

$$= \frac{3-0}{6-0}$$

$$= \frac{3}{6} = \frac{1}{2}$$

$$Log_4\, b = \tfrac{1}{2}$$

$$b = 4^{1/2}$$

$$b = \sqrt{4}$$

$$b = 2$$

$$Log_4\, a = 0$$

$$a = 4^0$$

$$a = 1$$

So $y = 2^x$

Now attempt Exercise 6 – 8 from the Logs & Exponentials booklet.