H Differentiation

SPTA Mathematics – Higher Notes

- There are 2 notations for Differentiation, conceived at almost the same time, 1670-1680 approx.:
  - Sir Isaac Newton: $\boldsymbol{f(x) = ax^n \implies f'(x) = anx^{n-1}}$
  - Gottfried Wilhelm Leibniz: $\boldsymbol{y = ax^n \implies \frac{dy}{dx} = anx^{n-1}}$

Always match the letters used.

where n is a rational number.

Leibniz Germany 1646 - 1716

Newton England 1643 - 1727

- y is the same as $f(x)$ and $\frac{dy}{dx}$ is the same as $f'(x)$.
- **The Derivative of a constant is zero.**
- **The Derivative of *ax* is *a*.**
- Leibniz notation is probably the most commonly used as its symbols relate precisely to the concepts being investigated – see later on!
- Both Newton and Leibniz had <u>**BISCUITS**</u> named after them, with Leibniz winning that battle too, Chocolate v Fig – you decide!!

Examples:

1. Differentiate the following:

a) $y = x^5$

$\implies \frac{dy}{dx} = 5x^4$

b) $g(x) = 4x^3$

$\implies g'(x) = 4 \times 3x^2$

$\implies g'(x) = 12x^2$

c) $y = 2m^{-4}$

$\implies \frac{dy}{dm} = 2 \times (-4)m^{-5}$

$\implies \frac{dy}{dm} = 8m^{-5}$

$\implies \frac{dy}{dm} = \frac{-8}{m^5}$

d) $f(p) = 5p^4 - 2p^3 + 7p^2 + 5p - 2$

$\implies f'(p) = 20p^3 - 6p^2 + 14p + 5$

Constant disappears

Now attempt Exercise 1 from the Differentiation booklet

- Prior to Differentiating a function the following must be true:
  - All brackets should be multiplied out – see Chain Rule later on in the course.
  - Roots need to be changed to powers: $\sqrt[m]{x^n} = x^{n/m}$
  - x cannot appear in the denominator of a fraction: $\frac{a}{bx^n} = \frac{a}{b}x^{-n}$
- After Differentiating it is good practice to return the expression to the form the question gave it in.

Examples:

2. Rearrange then differentiate the following:

a) $f(n) = \frac{1}{n^3}$

$\Longrightarrow f(n) = n^{-3}$

$\Longrightarrow f'(n) = 4 \times 3n^2$

$\Longrightarrow f'(n) = 12n^2$

b) $V = \frac{1}{7w^8}$

$\Longrightarrow V = \frac{1}{7}w^{-8}$

$\Longrightarrow \frac{dV}{dw} = \frac{1}{7} \times (-8)w^{-9}$

$\Longrightarrow \frac{dV}{dw} = \frac{-8}{7}w^{-9}$

$\Longrightarrow \frac{dV}{dw} = \frac{-8}{7w^9}$

c) $y = (2x+3)(x-2)$

$\Longrightarrow y = 2x^2 + 2x - 6$

$\Longrightarrow \frac{dy}{dx} = 4x + 2$

d) $y = \sqrt{x}$

$\Longrightarrow y = x^{1/2}$

$\Longrightarrow \frac{dy}{dx} = \frac{1}{2} \times x^{-1/2}$

$\Longrightarrow \frac{dy}{dx} = \frac{1}{2x^{1/2}}$

$\Longrightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{x}}$

e) $y = \sqrt[4]{x^5}$

$\Longrightarrow y = x^{5/4}$

$\Longrightarrow \frac{dy}{dx} = \frac{5}{4} \times x^{1/4}$

$\Longrightarrow \frac{dy}{dx} = \frac{5\sqrt[4]{x}}{4}$

f) $h(x) = \frac{3}{2\sqrt{x}}$

$\Longrightarrow h(x) = \frac{3}{2}x^{-1/2}$

$\Longrightarrow h'(x) = \frac{3}{2} \times -\frac{1}{2} \times x^{-3/2}$

$\Longrightarrow h'(x) = \frac{-3}{4x^{3/2}}$

$\Longrightarrow h'(x) = \frac{-3}{4\sqrt{x^3}}$

3. Differentiate the following:

Split into separate fractions and simplify before differentiating – do NOT bring the bottom up!!!!

a) $f(x) = \frac{x^4+2x^3}{3x}$

$\Longrightarrow f(x) = \frac{x^4}{3x} + \frac{2x^3}{3x}$

$\Longrightarrow f(x) = \frac{1}{3}x^3 + \frac{2}{3}x^4$

$\Longrightarrow f'(x) = 3 \times \frac{1}{3}x^2 + 4 \times \frac{2}{3}x^3$

$\Longrightarrow f'(x) = x^2 + \frac{8}{3}x^3$

b) $f(x) = \frac{x^3+3x^2-6x}{\sqrt{x}}$

$\Longrightarrow f(x) = \frac{x^3}{x^{1/2}} + \frac{3x^2}{x^{1/2}} - \frac{6x}{x^{1/2}}$

$\Longrightarrow f(x) = x^{5/2} + 3x^{3/2} - 6x^{1/2}$

$\Longrightarrow f'(x) = \frac{5}{2}x^{3/2} + \frac{3}{2} \times 3x^{1/2} - \frac{1}{2} \times 6x^{-1/2}$

$\Longrightarrow f'(x) = \frac{5}{2}x^{3/2} + \frac{9}{2}x^{1/2} - 3x^{-1/2}$

$\Longrightarrow f'(x) = \frac{5\sqrt{x^3}}{2} + \frac{9\sqrt{x}}{2} - \frac{3}{\sqrt{x}}$

c) $y = \sqrt{x}\left(x^2 + \sqrt[3]{x}\right)$

$\Longrightarrow y = x^{1/2}\left(x^2 + x^{1/3}\right)$

$\Longrightarrow y = x^{3/6}\left(x^{12/6} + x^{2/6}\right)$

$\Longrightarrow y = x^{15/6} + x^{5/6}$

$\Longrightarrow y = x^{5/2} + x^{5/6}$

$\Longrightarrow \frac{dy}{dx} = \frac{5}{2}x^{3/2} + \frac{5}{6}x^{-1/6}$

$\Longrightarrow \frac{dy}{dx} = \frac{5}{2}\sqrt{x^3} + \frac{5}{6x^{1/6}}$

$\Longrightarrow \frac{dy}{dx} = \frac{5}{2}\sqrt{x^3} + \frac{5}{6\sqrt[6]{x}}$

Now attempt Exercise 2 from the Differentiation booklet

Rate of Change:

- When we differentiate we are in fact calculating the RATE of CHANGE, i.e. how much one term changes in relation to another term .

- This is why Liebniz chose the notation: $\frac{dy}{dx} = \frac{difference\ in\ y}{difference\ in\ x} = \frac{change\ in\ y}{change\ in\ x}$ ← **How quickly y changes compared to x**

- The Rate of Change is found by differentiating the expression and then subbing in a value.
- There are a three special Rates of Change that we consider in Higher:
  - **GRADIENT of a CURVE** – This is the main use of Differentiation in Higher Maths:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{difference\ in\ y}{difference\ in\ x} = \frac{dy}{dx}$$

  - **Velocity** – The Velocity, ***v***, is defined as the rate of change of Displacement, ***s***, with respect to time, ***t*** : $v(t) = \frac{ds}{dt}$

Velocity ≈ Speed
Displacement ≈ Distance

  - **Acceleration** – The Acceleration, ***a***, is defined as the rate of change of Velocity, ***v***, with respect to time, ***t*** : $a(t) = \frac{dv}{dt}$

- Before substituting in a value for *x*:
  - Express all negative powers as positive powers: $\frac{a}{b}x^{-n} = \frac{a}{bx^n}$
  - Express all fractional powers as roots: $ax^{n/m} = a\sqrt[m]{x^n}$

Examples:

4. Calculate $f'(3)$ when $f(x) = 4x^6$:

$f'(x) = 6 \times 2x^5$ so $f'(3) = 12 \times (3)^5$

$f'(x) = 12x^5$ $f'(3) = 12 \times 243$

$f'(3) = 2916$

5. Calculate the Rate of Change of $y = \frac{4}{x^{2/3}}$, $(x \neq 0)$, when $x = 8$:

$$y = 4x^{-2/3}$$

$$\frac{dy}{dx} = \frac{-2}{3} \times 4x^{-5/3}$$

$$\frac{dy}{dx} = \frac{-8}{3x^{5/3}}$$

$$\frac{dy}{dx} = \frac{-8}{3\sqrt[3]{x^5}}$$

When $x = 8$

$$\frac{dy}{dx} = \frac{-8}{3\sqrt[3]{8^5}}$$

$$\frac{dy}{dx} = \frac{-8}{3(2^5)}$$

$$\frac{dy}{dx} = \frac{-8}{3(32)}$$

$$\frac{dy}{dx} = \frac{-8}{96} = \frac{-1}{12}$$

6. Calculate the Gradient of the Curve $y = \frac{9}{x^3}$, $(x \neq 0)$, where $x = -3$:

$$y = 9x^{-3}$$

$$\frac{dy}{dx} = -3 \times 9x^{-4}$$

$$\frac{dy}{dx} = \frac{-27}{x^4}$$

When $x = -3$

$$\frac{dy}{dx} = \frac{-27}{(-3)^4}$$

$$\frac{dy}{dx} = \frac{-27}{81}$$

$$\frac{dy}{dx} = \frac{-1}{3} \qquad \text{so } m = \frac{-1}{3}$$

7. Find the points on the Curve $f(x) = x^3 - 3x^2 + 8x - 5$ where the Gradient is 8

Points = Coordinates, 2 or more!

Gradient = Derivative

$$f'(x) = 3x^2 - 6x + 8$$

When $m = 8$

$$3x^2 - 6x + 8 = 8$$

$$3x^2 - 6x = 0$$

$$3x(x - 2) = 0$$

$$3x = 0 \text{ or } x - 2 = 0$$

$$x = 0 \text{ or } x = 2$$

y is the same as $f(x)$

When $x = 0$: $y = 0^3 - 3(0)^2 + 8(0) - 5$

$y = -5$ so Pt(0 , –5)

When $x = 2$: $y = 2^3 - 3(2)^2 + 8(2) - 5$

$y = 8 - 12 + 16 - 5$

$y = 7$ so Pt(0 , 7)

8. **a)** A ball is thrown in the air so that its displacement, s, metres after t seconds is given by $s(t) = 45t - 5t^2$. Find its velocity after 3 seconds.

Derivative of Displacement

$s'(t) = 45 - 10t$ When $t = 3$ $s'(3) = 45 - 10(3)$

$s'(3) = 45 - 30$

$s'(3) = 15$ so v = 15 m/ sec

b) After how many seconds will the ball come to a rest?

When $v = 0$: $45 - 10t = 0$

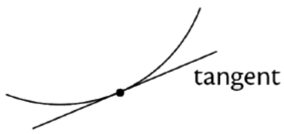
$10t = 45$

$t = 4.5$ So the ball will come to a rest after 4.5 seconds

Comes to a rest when Velocity is zero.

Now attempt Exercise 3A/B from the Differentiation booklet

Equation of a Tangent:



- A Tangent is a Straight Line which touches a Curve at one point only.
- For this to happen the Gradient of the Curve and the Tangent's Gradient at the point of contact must be the same.
- So to find the Equation of the Tangent do the following:
  - Differentiate the curve and substitute in the x-coordinate to find the Gradient.
  - Find the y-coordinate by subbing x into the original equation (y may be given!!)
  - Use $y - b = m(x - a)$ to find the equation of the tangent.

Examples:

9. Find the Equation to the Tangent for the following curves:

a) $y = 2x^2 + 3x - 6$, at the point $(2, 8)$

$\frac{dy}{dx} = 4x + 3$ $\qquad$ $y - b = m(x - a)$

When $x = 2$ $\qquad$ $\frac{dy}{dx} = 4(2) + 3$ $\qquad$ $y - 8 = 11(x - 2)$

$\frac{dy}{dx} = 8 + 3$ $\qquad$ $y - 8 = 11x - 22$

$\boxed{\text{GRADIENT}} \rightarrow \frac{dy}{dx} = 11$ $\qquad$ $y = 11x - 14$

b) $y = 6\sqrt[3]{x^2} + 12$, when $x = -8$

$y = 6x^{2/3} + 12$

$\frac{dy}{dx} = \frac{2}{3} \times 6x^{-1/3}$

$\frac{dy}{dx} = \frac{4}{\sqrt[3]{x}}$

When $x = -8$ $\qquad$ $\frac{dy}{dx} = \frac{4}{\sqrt[3]{8}}$

$\frac{dy}{dx} = \frac{4}{2}$

$\frac{dy}{dx} = 2$

$\boxed{\text{Sub } x \text{ into original to find } y}$

$y = 6\sqrt[3]{8^2} + 12$

$y = 6 \times 2^2 + 12$

$y = 36$

So $m = 2$ and the Pt$(-8, 36)$

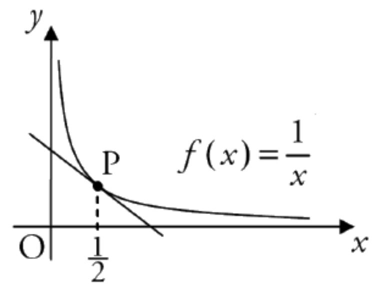
$y - b = m(x - a)$

$y - 36 = 2(x + 8)$

$y - 36 = 2x + 16$

$y = 2x + 52$

10. Find the equation of the tangent to the curve at point P:



$f(x) = x^{-1}$

$f'(x) = -1 \times x^{-2}$

$f'(x) = \frac{-1}{x^2}$

When $x = ½$

$f'(x) = \frac{-1}{x^2}$

$f'(x) = \frac{-1}{({}^1/_2)^2}$

$f'(x) = -1 \div \frac{1}{4}$

$f'(x) = -1 \times \frac{4}{1}$

$f'(x) = -4$

$y = \frac{1}{{}^1/_2}$

$y = 1 \div \frac{1}{2}$

$y = 1 \times \frac{2}{1}$

$y = 2$

so $m = -4$ and Pt $P(½ , 2)$

$y - b = m(x - a)$

$y - 2 = -4(x - ½)$

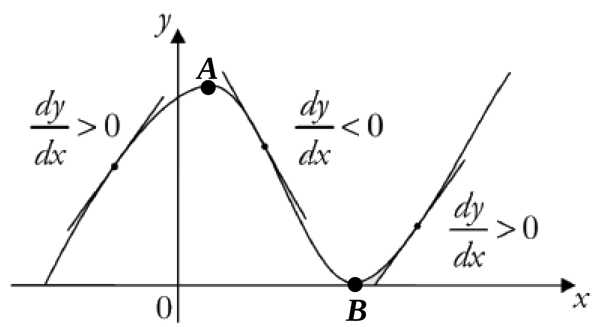
$y - 2 = -4x + 2$

$y = -4x + 4$

Now attempt Exercise 4 from the Differentiation booklet

Increasing/Decreasing/Stationary:

- The Derivative can be used to determine the shape of the curve.
- If the Gradient of the Curve (Derivative) is positive the curve is said to be **INCREASING** , $\frac{dy}{dx} > 0$
- If the Gradient of the Curve (Derivative) is negative the curve is said to be **DECREASING** , $\frac{dy}{dx} < 0$
- This can be seen clearer on a diagram:



- Can parts of the curve be neither Increasing nor Decreasing? These points are called **STATIONARY POINTS** and occur when the Gradient is ZERO, i.e. when $\frac{dy}{dx} = 0$.
- So in the Diagram above points A and B are stationary and have $\frac{dy}{dx} = 0$.

Examples:

11. Find the intervals for which the function $y = x^2 - 6x - 6$ is:

a) Increasing

$\frac{dy}{dx} = 2x - 6$

Increasing when $\frac{dy}{dx} > 0$

$2x - 6 > 0$

$2x > 6$

$x > 3$

(b) Decreasing

Decreasing when $\frac{dy}{dx} < 0$

$2x - 6 < 0$

$2x < 6$

$x < 3$

(c) Stationary

Stationary when $\frac{dy}{dx} = 0$

$2x - 6 = 0$

$2x = 6$

$x = 3$

12. Determine whether or not the curves are increasing, decreasing or stationary at the given point:

a) $f(x) = 2x^3 - 5x^2 + 8$ When $x = 3$

$f'(x) = 6x^2 - 10x$

$f'(3) = 6(3)^2 - 10(3)$

$f'(3) = 54 - 30$

$f'(3) = 24$ Since $\frac{dy}{dx} > 0$ the function is increasing when x = 3.

b) $y = \frac{6}{\sqrt[3]{x^2}} + 12x$ When $x = -8$

$y = 6x^{-\frac{2}{3}} - 12x$

$\frac{dy}{dx} = -\frac{2}{3} \times 6x^{-\frac{5}{3}} - 12$

$\frac{dy}{dx} = -\frac{4}{\sqrt[3]{x^5}} - 12$

$\frac{dy}{dx} = -\frac{4}{\sqrt[3]{-8^5}} - 12$

$\frac{dy}{dx} = \frac{1}{8} - 12 = -\frac{95}{8}$ Since $\frac{dy}{dx} < 0$ the function is decreasing when x = −8.

c) Prove that the function $y = x^3 - 6x^2 + 12x - 7$ is never decreasing:

$$\frac{dy}{dx} = 3x^2 - 12x + 12$$

$$\frac{dy}{dx} = 3(x^2 - 4x + 4)$$

$$\frac{dy}{dx} = 3(x - 2)(x - 2)$$

$$\frac{dy}{dx} = 3(x - 2)^2$$

Since $\frac{dy}{dx} \geq 0$ for all values of x the function will never be decreasing.

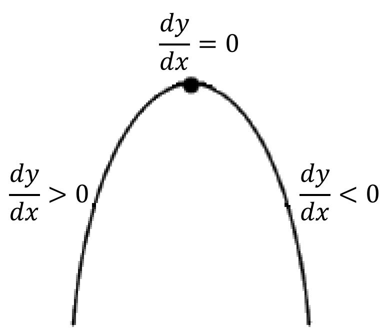
Squared term is always positive, so: + x + = +

Now attempt Exercise 5 from the Differentiation booklet

Nature (Type) of Stationary Points:

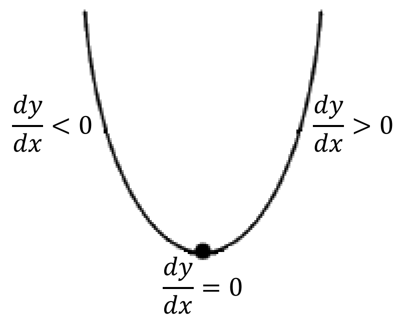
- Stationary points occur when $\frac{dy}{dx} = 0$. **Will need to be written a lot!!**
- You can determine a Stationary Point's Nature using a **NATURE TABLE**.
- A Nature Table looks at the values of Curve's Gradients to determine its shape.
- There are 4 types of Stationary Points as shown below:
  - 2 Turning Points:

Maximum Turning Point



x	→	x	→
$^{dy}/_{dx}$	+	0	–
Shape	/	—	\

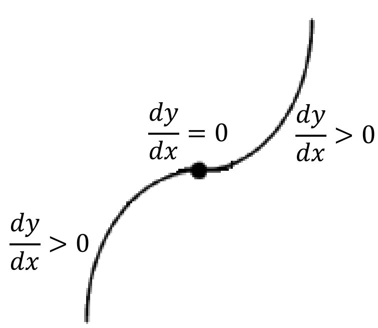
Minimum Turning Point



x	→	x	→
$^{dy}/_{dx}$	–	0	+
Shape	\	_	/

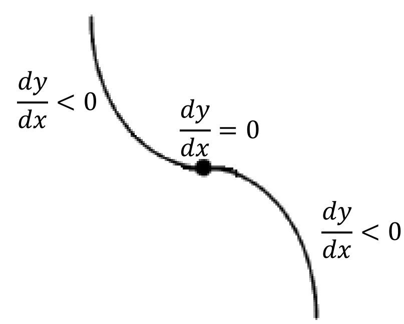
  - 2 Points of Inflection:

Positive Point of Inflection



x	→	x	→
$^{dy}/_{dx}$	+	0	+
Shape	/	—	/

Negative Point of Inflection



x	→	x	→
$^{dy}/_{dx}$	–	0	–
Shape	\	—	\

- Stationary and Turning Points mean the same thing for our purposes and can be interchanged!
- We only need to know the x-coordinate of the Stationary Point to determine its Nature but usually we find the y-coordinate by subbing x into the original Function.
- We can use the Nature Table to state the range of values a function is Increasing/Decreasing.

Examples:

13. a) Find the stationary points on the curve, $y = x^3 - 6x^2 + 9x - 4$ and determine their Nature:

$\frac{dy}{dx} = 3x^2 - 12x + 9$

$\frac{dy}{dx} = 3(x^2 - 4x + 3)$

$\frac{dy}{dx} = 3(x-3)(x-1)$

Stationary Point occur when $\frac{dy}{dx} = 0$ so

Sentence MUST be written!!

$3(x-3)(x-1) = 0$

$(x-3)(x-1) = 0$

$x - 3 = 0$ or $x - 1 = 0$

$x = 3$ or $x = 1$

When $x = 1$:

$y = (1)^3 - 6(1)^2 + 9(1) - 4$

$y = 1 - 6 + 9 - 4$

$y = 0$ so Pt(1 , 0)

When $x = 3$:

$y = (3)^3 - 6(3)^2 + 9(3) - 4$

$y = 27 - 54 + 27 - 4$

$y = -4$ so Pt(3 , –4)

Nature Table

List the SP in order, smallest to largest.

x	→	1	→	→	3	→
$^{dy}/_{dx}$	+	0	–	–	0	+
Shape	/	—	\	\	_	/

When $x = 0$ $\frac{dy}{dx} = 9$ so Positive

When $x = 2$ $\frac{dy}{dx} = -3$ so Negative

When $x = 4$ $\frac{dy}{dx} = 9$ so Positive

So Max TP at (1 , 0) & Min TP at (3 , –4)

Pick values before, in between and after the stationary points. You do NOT need to show this working unless you want to.

b) (i) State the range of values for which the function is strictly increasing:

$x < 1$ & $x > 3$

(ii) State the range of values for which the function is strictly decreasing:

$1 < x < 3$

(iii) State the range of values for which the function is not increasing:

$1 \leq x \leq 3$

NOT Increasing so Decreasing & Stationary

Now attempt Exercise 6 from the Differentiation booklet

Sketching Curves:

- To sketch a curve given its equation you do the following:
  - Find the Stationary Points and Determine their Nature.
  - Find the y-intercept by making $x = 0$
  - Find its roots by making $y = 0$
  - Sketch and fully annotate the curve.

First 3 steps can be done in any order!!

Examples:

14. a) Sketch the curve, $f(x) = 8x^3 - 3x^4$:

$f'(x) = 24x^2 - 12x^3$

$f'(x) = 12x^2(2 - x)$

Stationary Point occur when $f'(x) = 0$ so

$12x^2(2 - x) = 0$

$12x^2 = 0$ or $2 - x = 0$

$x = 0$ or $x = 2$

When $x = 0$:

$f(x) = 8 \times (0)^3 - 3 \times (0)^4$

$f(x) = 0$ so Pt $(0, 0)$

When $x = 2$:

$f(x) = 8 \times (2)^3 - 3 \times (2)^4$

$f(x) = 64 - 48$

$f(x) = 16$ so Pt $(2, 16)$

Nature Table

x	$\rightarrow$	0	$\rightarrow$	$\rightarrow$	2	$\rightarrow$
$f'(x)$	+	0	+	+	0	+
Shape	/	—	/	/	—	\

When $x = -1$ $f'(x) = 36$ so Positive

When $x = 1$ $f'(x) = 12$ so Positive

When $x = 3$ $f'(x) = -108$ so Negative

Pick values before, in between and after the stationary points. You do NOT need to show this working unless you want to.

So +'ve PI at $(0, 0)$ & Max TP at $(2, 16)$

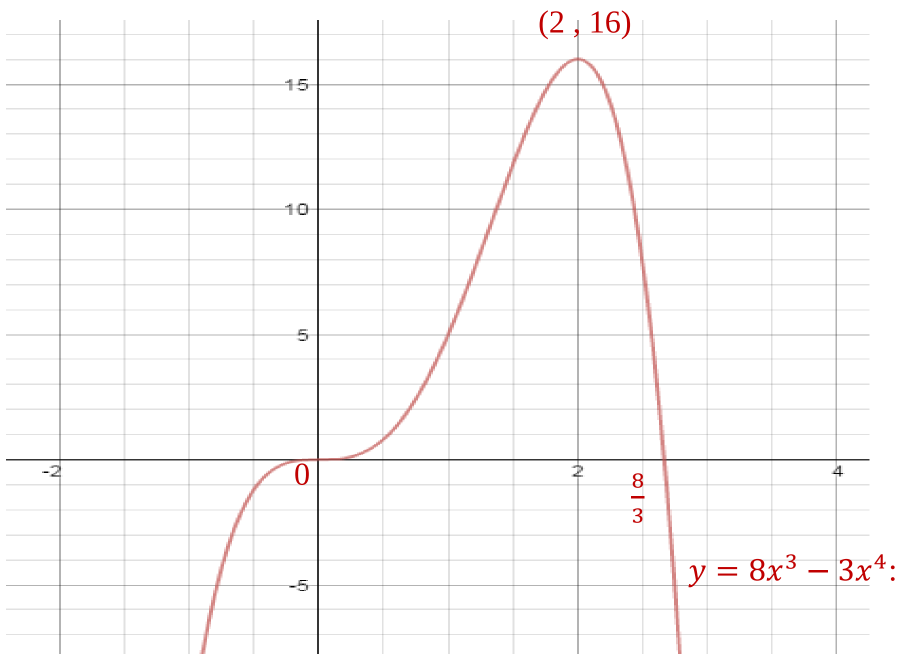
Cuts y-axis when $x = 0$: $f(x) = 8 \times (0)^3 - 3 \times (0)^4$

$f(x) = 0$ so Pt $(0, 0)$

Cuts x-axis when $f(x) = 0$: $x^3(8 - 3x) = 0$

$x^3 = 0$ or $8 - 3x = 0$

$x = 0$ or $x = \frac{8}{3}$ so Pt $(0, 0)$ & Pt $(\frac{8}{3}, 0)$



b) (i) State the range of values for which the function is strictly increasing:

$0 < x < 2$

(ii) State the range of values for which the function is strictly decreasing:

$x < 0 \ \& \ x > 2$

Now attempt Exercise 7 from the Differentiation booklet

Max/Min in a Closed Interval:

- A closed interval is when the range of values is limited, e.g. $-2 \leq x \leq 8$:
- To find the Max/Min points in a Closed Interval do the following:
  - Find the Stationary Points – No Need for a Nature Table.
  - Find the coordinates of the 2 end points of the interval.
  - State the Max/Min y value and their corresponding values of x.

First 2 steps can be done in any order!!

Examples:

15. Find the Max & Min values of the curve $y = 2x^3 - 5x^2 - 4x + 1$ on the interval $-1 \leq x \leq 4$ and their corresponding values of x.

$\frac{dy}{dx} = 6x^2 - 10x - 4$

$\frac{dy}{dx} = (3x + 1)(2x - 4)$

Stationary Point occur when $\frac{dy}{dx} = 0$ so

$(3x + 1)(2x - 4) = 0$

$3x = -1$ or $2x = 4$

$x = -\frac{1}{3}$ or $x = 2$

When $x = -\frac{1}{3}$:

$y = 2(\frac{-1}{3})^3 - 5(\frac{-1}{3})^2 - 4(\frac{-1}{3}) + 1$

$y = \frac{-2}{27} - \frac{5}{9} + \frac{4}{3} + 1$

$y = \frac{46}{27}$ so Pt$(-\frac{1}{3}, \frac{46}{27})$

When $x = 2$:

$y = 2(2)^3 - 5(2)^2 - 4(2) + 1$

$y = 16 - 20 - 8 + 1$

$y = -11$ so Pt$(2, -11)$

There is no need to do a Nature Table!!

<u>End Points</u>

When $x = -1$:

$y = 2(-1)^3 - 5(-1)^2 - 4(-1) + 1$

$y = -2 - 5 + 4 + 1$

$y = -2$ so Pt$(-1, -2)$

When $x = 4$:

$y = 2(4)^3 - 5(4)^2 - 4(4) + 1$

$y = 128 - 80 - 16 + 1$

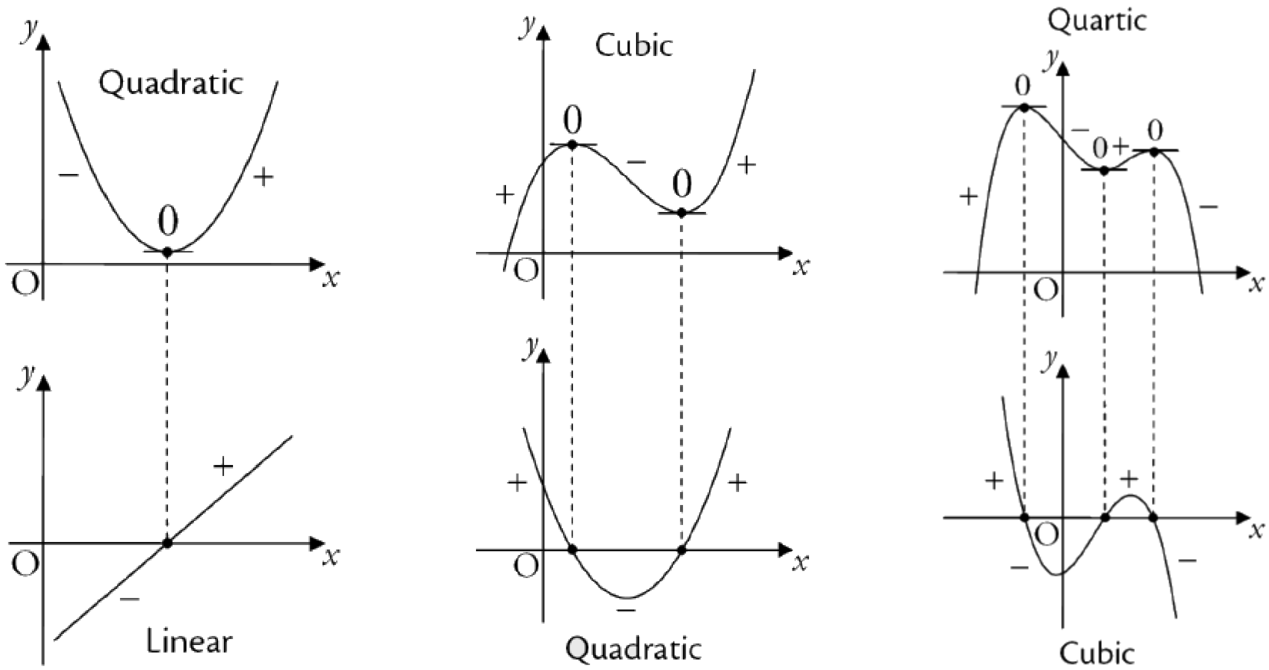
$y = 33$ so Pt$(4, 33)$

So Max = 33 when $x = 4$ and Min = -11 when $x = 2$

Now attempt Exercise 8 from the Differentiation booklet

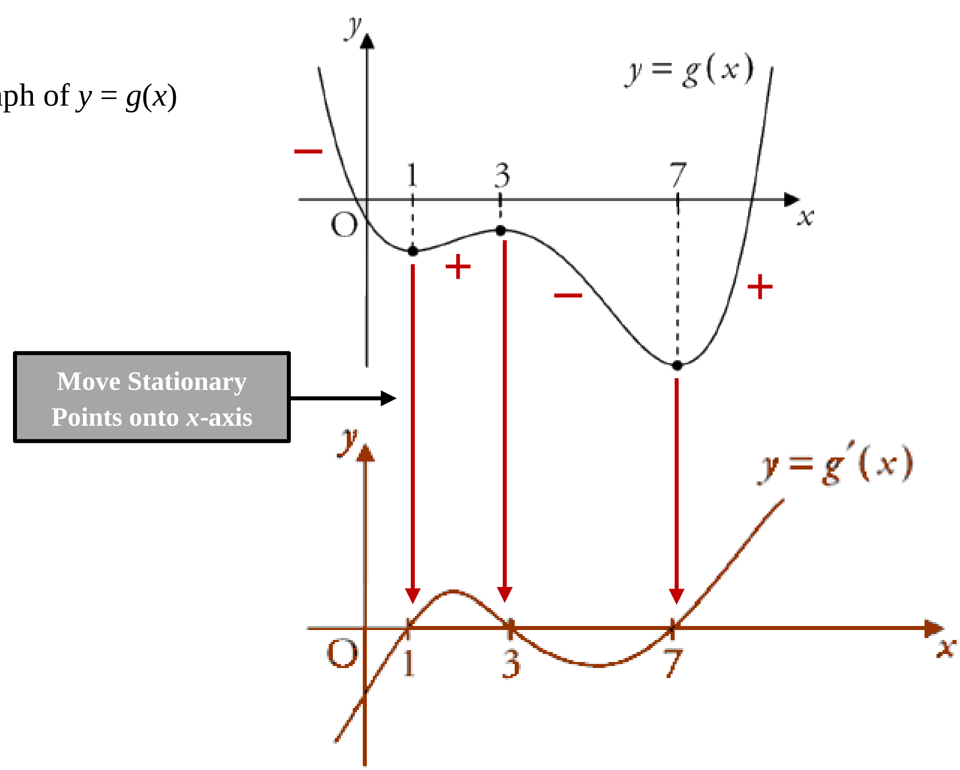
Graphs of Derived Functions:

- We can sketch the Graph of a Derived function by looking at the Gradient at each point on the original curve:
  - Stationary Points move onto the x-axis.
  - Negaitive Gradient is below x-axis and Positive Gradient is above x-axis
- The Degree of the Derived Graph will always be one less than the original.



Examples:

16. Sketch the Derived Graph of $y = g(x)$



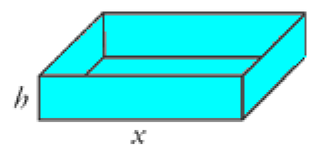
Now attempt Exercise 9 from the Differentiation booklet

Optimisation:

- Max/Min in a Closed Interval is one type of Optimisation.
- Optimisation is the process of finding the Maximum or Minimum value of a function subject to some condition/constraint.
- Optimisation is usually expressed in a real-life scenario and the answer must always be in context.
- Optimisation questions normally come in 2 parts:
  - A Show that question involving the manipulation of functions.
  - A Differentiation question to find the Max/Min.
  - Even if you **cannot** do the first part you should still be able to do the second, in fact most teachers tell their students to do the second part first as this is worth more marks and usually more straight forward to do!!

Examples:

17. A plastic tray (open topped) has volume 108cm^3.
It has a square base of length x cm and height h cm.



a) Show that the surface area, A, of the tray is given by:
$A(x) = x^2 + \frac{432}{x}$

$A(x) = 4\times\text{Area of Sides}+\text{Area of base}$

$A(x) = 4\times(x \times h) + x \times x$

$A(x) = 4xh + x^2$

***h* does not appear in the SA Function above. Still to use Volume = 108!!**

$V = L \times B \times H$

$108 = x \times x \times h$

$108 = x^2 \times h$

$h = \frac{108}{x^2}$

$A(x) = 4x\left(\frac{108}{x^2}\right) + x^2$

$A(x) = \frac{432x}{x^2} + x^2$

$A(x) = \frac{432}{x} + x^2$ as required

b) Find the dimensions of the tray which uses the smallest amount of plastic.

$A(x) = 432x^{-1} + x^2$

$A'(x) = -432x^{-2} + 2x$ Stationary Point occur when $A'(x) = 0$ so

$A'(x) = -\frac{432}{x^2} + 2x$ $-\frac{432}{x^2} + 2x = 0$

$$2x = \frac{432}{x^2}$$

$$2x^3 = 432$$

$$x^3 = 216$$

$$x = 6$$

Nature Table

x	$\rightarrow$	6	$\rightarrow$
$A'(x)$	–	0	+
Shape	\	__	/

When $x = 5$ $A'(x) = -7.28$ so Negative

When $x = 7$ $A'(x) = 5.18$ so Positive

No need to show this

So Minimum amount of plastic when $x = 6$ cm

- Sometimes the question will only ask you to "Find the value of x which minimises the Area". We would therefore stop here!
- You may be asked for the dimensions as we are in this question, so we calculate L, B & H.

Dimensions

Dimensions: Length = 6 cm, Breadth = 6 cm and $h = \frac{108}{x^2}$

$$h = \frac{108}{6^2}$$

$$h = \frac{108}{36}$$

$$h = 3 \text{ cm}$$

- You may be asked for the Minimum area, so sub in x to the given formula:

When $x = 6$:

$$A(6) = \frac{432}{6} + 6^2$$

$$A(6) = 72 + 36$$

$$A(6) = 108 \text{ cm}^2$$

Now attempt Exercises 10 & 11 from the Differentiation booklet