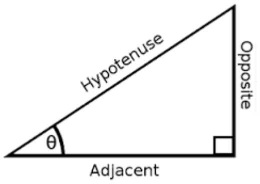
H Compound Angle

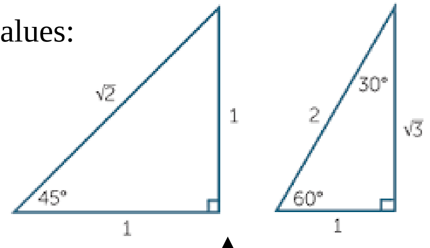
SPTA Mathematics – Higher Notes

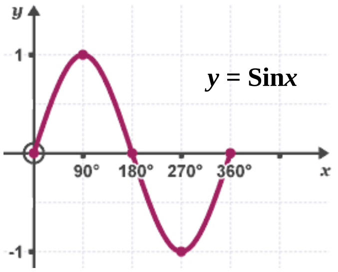
Trig Stuff from Nat 5 & earlier in the Higher:

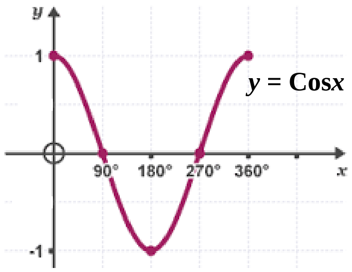
- SOHCAHTOA: $\sin\theta = \frac{opp}{hyp}$ $\cos\theta = \frac{adj}{hyp}$ $\tan\theta = \frac{opp}{adj}$



- Exact Values:

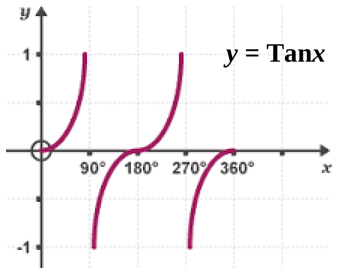






Use the TRIANGLES for exact values of 30, 45 & 60

Use the GRAPHS for exact values of 0, 90, 180, 270 & 360

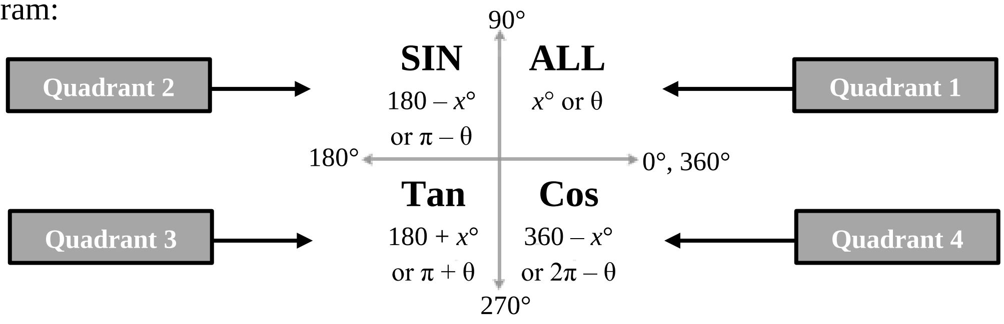


- Radians: π radians = 180°

  Convert to RADIANS by dividing by 180 and multiplying by π

  Convert to DEGREES by dividing by π and multiplying by 180

- CAST Diagram:



- Solving Trig Equations: You should already be able to solve Trig Equations in the form $y = a\sin(bx + c) + d$ or $y = a\cos(bx + c) + d$

- Trig Identities: $\tan\theta = \frac{\sin\theta}{\cos\theta}$ and $\sin^2 x + \cos^2 x = 1$

Compund Angle Formulae:

- Also known as the **ADDITION FORMULAE**.
- There are 4 Compound Angle Formulae as follows:

1. $\sin(A+B) = \sin A \cos B + \cos A \sin B$

2. $\sin(A-B) = \sin A \cos B - \cos A \sin B$

3. $\cos(A+B) = \cos A \cos B - \sin A \sin B$

4. $\cos(A-B) = \cos A \cos B + \sin A \sin B$

How they appear on the SQA Formula List

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

Careful!! The signs are flipped!!!

Examples:

1. Expand and simplify the following:

a) $\sin(180+y) = \sin 180 \cos y + \cos 180 \sin y$

$= (0)\cos y + (-1)\sin y$

$= -\sin y$

b) $\cos\left(x + {}^{\pi}/_{2}\right) = \cos x \cos {}^{\pi}/_{2} - \sin x \sin {}^{\pi}/_{2}$

$= (0)\cos x - (1)\sin x$

$= -\sin x$

Use the GRAPHS to find the exact values of 0, 90 & 180

c) $\cos(-p) = \cos(0-p)$

$= \cos 0 \cos p + \sin 0 \sin p$

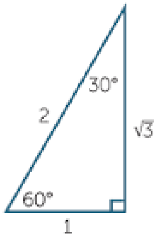
$= (1)\cos p + (0)\sin p$

$= \cos p$

d) $\sin(30-y) = \sin 30 \cos y - \cos 30 \sin y$

$= \left(\frac{1}{2}\right)\cos y - \left(\frac{\sqrt{3}}{2}\right)\sin y$

$= \frac{1}{2}\cos y - \frac{\sqrt{3}}{2}\sin y$



Note: In part (b) above you can change the radians to degrees if you find this easier.

2. Simplify the following:

a) $\cos 130 \cos 50 - \sin 130 \sin 50 = \cos(130 + 50)$

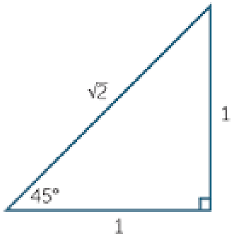
$= \cos(180)$

$= -1$

b) $\sin 145 \cos 100 - \cos 145 \sin 100 = \sin(145 - 100)$

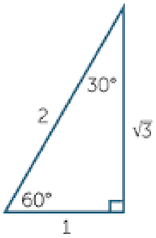
$= \sin(45)$

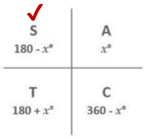
$= \frac{1}{\sqrt{2}}$



3. Prove that: $\sin(60 + y) + \cos(y - 150) = \sin y$

$\sin(60 + y) + \cos(y + 150) = \sin 60 \cos y + \cos 60 \sin y + \cos y \cos 150 + \sin y \sin 150$





sin150 = sin30

cos150 = −cos30

$= {}^{\sqrt{3}}/_{2} \cos y + {}^{1}/_{2} \sin y + \cos y \left(-{}^{\sqrt{3}}/_{2}\right) + \sin y \left({}^{1}/_{2}\right)$

$= {}^{\sqrt{3}}/_{2} \cos y + {}^{1}/_{2} \sin y - {}^{\sqrt{3}}/_{2} \cos y + {}^{1}/_{2} \sin y$

$= {}^{\sqrt{3}}/_{2} \cos y - {}^{\sqrt{3}}/_{2} \cos y + {}^{1}/_{2} \sin y + {}^{1}/_{2} \sin y$

$= \sin y$ Hence proven.

4. Show that $\sin(A + B) + \sin(A - B) = 2 \sin A \cos B$

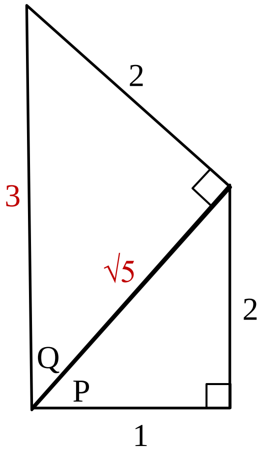
$\sin(A + B) + \sin(A - B) = \sin A \cos B + \cos A \sin B + \sin A \cos B - \cos A \sin B$

$= 2 \sin A \cos B$ as required

Now attempt Exercise 1 from the Compound Angle Formulae booklet

5. Calculate the exact value of $\sin(P + Q)$**:**

Use Pythagoras to find the missing sides in the triangles, then use SOHCAHTOA



$\sin P = \frac{2}{\sqrt{5}}$ $\sin Q = \frac{2}{3}$

$\cos P = \frac{1}{\sqrt{5}}$ $\cos Q = \frac{\sqrt{5}}{3}$

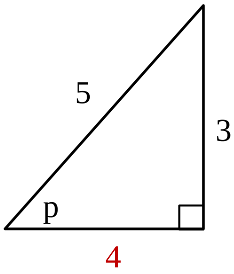
$\sin(P + Q) = \sin P \cos Q + \cos P \sin Q$

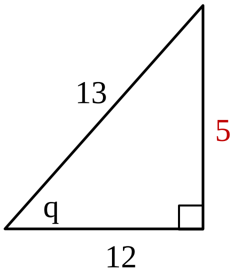
$= \frac{2}{\sqrt{5}} \times \frac{\sqrt{5}}{3} + \frac{1}{\sqrt{5}} \times \frac{2}{3}$

$= \frac{2\sqrt{5}}{3\sqrt{5}} + \frac{2}{3\sqrt{5}}$

$= \frac{2\sqrt{5}+2}{3\sqrt{5}}$

6. Given that $\sin p = \frac{3}{5}$ and $\cos q = \frac{12}{13}$ prove that $\cos(p-q) = \frac{63}{65}$





$\cos p = \frac{4}{5}$

$\sin q = \frac{5}{13}$

Use Pythagoras to find the missing sides in the triangles, then use SOHCAHTOA

$$\cos(p-q) = \cos p \cos q + \sin p \sin q$$

$$= \frac{4}{5} \times \frac{12}{13} + \frac{3}{5} \times \frac{5}{13}$$

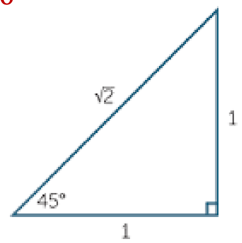
$$= \frac{48}{65} + \frac{15}{65}$$

$$= \frac{63}{65} \text{ as required.}$$

7. By expressing 75° = 45° + 30° find the exact value of sin75°

$$\sin 75 = \mathrm{Sin}(45+30) \;= \sin 45 \cos 30 + \cos 45 \sin 30$$

$$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \times \frac{1}{2}$$



$$= \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}}$$

$$= \frac{\sqrt{3}+1}{2\sqrt{2}}$$

This type of question may ask you to express your answer with a RATIONAL DENOMINATOR. This is from National 5 and only needs to be done if stated in the question. It is done as follows:

$$= \frac{\sqrt{3}+1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$

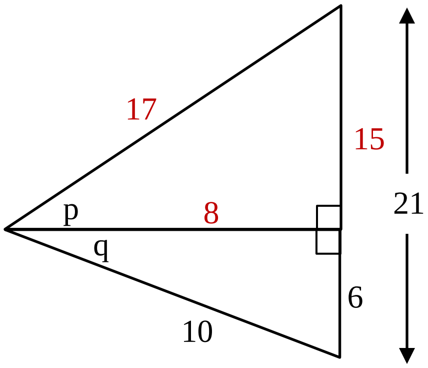
$$= \frac{\sqrt{6}+\sqrt{2}}{2\sqrt{4}}$$

$$= \frac{\sqrt{6}+\sqrt{2}}{4}$$

8. Using the diagram opposite show that:

a) **(i)** $\sin(p+q) = \frac{84}{85}$ **(ii)** $\cos(p+q) = -\frac{13}{85}$

b) Hence find the exact value of $\tan(p+q)$



a) **(i)** $\sin p = \frac{15}{17}$ $\cos p = \frac{8}{17}$

$\sin q = \frac{6}{10} = \frac{3}{5}$ $\cos q = \frac{8}{10} = \frac{4}{5}$

$$\sin(p+q) = \sin p \cos q + \cos p \sin q$$

$$= \frac{15}{17} \times \frac{4}{5} + \frac{8}{17} \times \frac{3}{5}$$

$$= \frac{60}{85} + \frac{24}{85}$$

$$= \frac{84}{85} \text{ as required.}$$

(ii) $\cos(p+q) = \cos p \cos q - \sin p \sin q$

$$= \frac{8}{17} \times \frac{4}{5} - \frac{15}{17} \times \frac{3}{5}$$

$$= \frac{32}{85} - \frac{45}{85}$$

$$= -\frac{13}{85} \text{ as required.}$$

b) $\tan\theta = \frac{\sin\theta}{\cos\theta}$ so $\tan(p+q) = \frac{\sin(p+q)}{\cos(p+q)}$

$$= \frac{84}{85} \div \frac{-13}{85}$$

$$= \frac{84}{85} \times \frac{85}{-13}$$

$$= \frac{84}{-13}$$

$$= -\frac{84}{13} \text{ or } -6\frac{6}{13}$$

Unless stated in the question you can leave your answer as either a Top Heavy Fraction or a Mixed Number.

Now attempt Exercise 2 from the Compound Angle Formulae booklet

Double Angle Formulae:

- There are also 4 Double Angle Formulae as follows:

1. $\sin 2A = 2\sin A\cos A$

2. $\cos 2A = \cos^2 A - \sin^2 A$

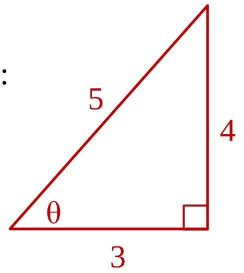
3. $\cos 2A = 2\cos^2 A - 1$

4. $\cos 2A = 1 - 2\sin^2 A$

ALL 4 are given on the SQA Formula List

Examples:

9. Given that $\tan\theta = \frac{4}{3}$ calculate the exact value of:



$\sin\theta = \frac{4}{5}$

$\cos\theta = \frac{3}{5}$

a) sin2θ: $\sin 2\theta = 2\sin\theta\cos\theta$

$$= 2 \times \frac{4}{5} \times \frac{3}{5}$$

$$= \frac{24}{25}$$

b) cos2θ: $\cos 2\theta = 2\cos^2\theta - 1$

$$= 2 \times \left(\frac{3}{5}\right)^2 - 1$$

$$= 2 \times \frac{9}{25} - 1$$

$$= \frac{18}{25} - 1$$

$$= \frac{18}{25} - \frac{25}{25} = -\frac{7}{25}$$

You can use any of the 3 formulae to get the same answer!!

$\cos 2\theta = 1 - 2\sin^2\theta$

$$= 1 - 2 \times \left(\frac{4}{5}\right)^2$$

$$= 1 - 2 \times \frac{16}{25}$$

$$= 1 - \frac{32}{25}$$

$$= \frac{25}{25} - \frac{32}{25} = -\frac{7}{25}$$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$$= \left(\frac{3}{5}\right)^2 - \left(\frac{4}{5}\right)^2$$

$$= \frac{9}{25} - \frac{16}{25} = -\frac{7}{25}$$

c) tan2θ: $\tan\theta = \frac{\sin\theta}{\cos\theta}$ so $\tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta}$

$$= \frac{24}{25} \div \frac{-7}{25}$$

$$= \frac{24}{25} \times \frac{25}{-7}$$

$$= \frac{24}{-7} \text{ or } -2\frac{3}{7}$$

d) cos4θ: $\cos 4\theta = \cos^2 2\theta - \sin^2 2\theta$

$$= \left(-\frac{7}{25}\right)^2 - \left(\frac{24}{25}\right)^2$$

$$= \frac{49}{625} - \frac{576}{625} = -\frac{527}{625}$$

Use your answers from part (a) & (b)

When asked to find cos2θ you can use <u>ANY</u> of the 3 formulae expressed above and always get the same answer!!

10. Given that $\cos\theta = \frac{2}{\sqrt{5}}$ calculate the exact value of cos2θ

$$\cos 2\theta = 2\cos^2\theta - 1$$

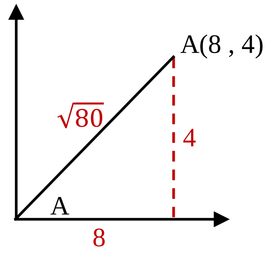
$$= 2 \times \left(\frac{2}{\sqrt{5}}\right)^2 - 1$$

$$= 2 \times \frac{4}{5} - 1$$

$$= \frac{8}{5} - 1$$

$$= \frac{8}{3} - \frac{5}{5} = \frac{3}{5}$$

11. For the diagram shown find the exact values of:



$$\sin A = \frac{4}{\sqrt{80}} = \frac{4}{4\sqrt{5}} = \frac{1}{\sqrt{5}}$$

$$\cos A = \frac{8}{\sqrt{80}} = \frac{2}{\sqrt{5}}$$

a) sin2A: $\sin 2A = 2\sin A\cos A$

$$= 2 \times \frac{1}{\sqrt{5}} \times \frac{2}{\sqrt{5}}$$

$$= \frac{4}{5}$$

b) cos2A: $\cos 2A = 1 - 2\sin^2 A$

$$= 1 - 2 \times \left(\frac{1}{\sqrt{5}}\right)^2$$

$$= 1 - 2 \times \frac{1}{5}$$

$$= 1 - \frac{2}{5} = \frac{3}{5}$$

c) By expressing $\cos 3A$ as $\cos(2A + A)$ find the exact value of $\cos 3A$ expressing your answer with a rational denominator:

$$\cos 3\mathrm{A} = \cos(2A + A) = \cos 2A\cos A - \sin 2A\sin A$$

$$= \frac{3}{5} \times \frac{2}{\sqrt{5}} - \frac{4}{5} \times \frac{1}{\sqrt{5}}$$

$$= \frac{6}{5\sqrt{5}} - \frac{4}{5\sqrt{5}}$$

$$= \frac{2}{5\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}} \quad \rightarrow \quad = \frac{2\sqrt{5}}{5\times5} = \frac{2\sqrt{5}}{25}$$

Now attempt Exercise 3 from the Compound Angle Formulae booklet

Solving Trig Equations:

- We already know how to solve Trig equations in the form $y = a\sin 2x + c$, or similar.
- We don't yet know how to solve equations with a mixture of values of x or trig, like those below.
- The Double Angle Formulae along with Factorisation can be used to solve equations in these forms:
    - $a\sin 2x + b\sin x = 0$ or $a\sin 2x + b\cos x = 0$ by expanding **<u>sin2*x*</u>** and then taking out a Common factor.
    - $y = a\cos 2x + b\sin x + c$ or $y = a\cos 2x + b\cos x + c$ by expanding **<u>cos2*x*</u>** and then using Quadratic Factorisation.
- Remember from earlier:

1. $\sin 2A = 2\sin A\cos A$

Sin2x only has 1 possibility

2. ~~$\cos 2A = \cos^2 A - \sin^2 A$~~

Don't use $\cos^2 x - \sin^2 x$ to replace cos2*x*

3. $\cos 2A = 2\cos^2 A - 1$

4. $\cos 2A = 1 - 2\sin^2 A$

Look at the other Trig parts of the equation to decide which you use!!

Examples:

12. Solve $2\sin 2x + 3\sin x = 0$ where $0 \le x \le 180°$

$$2\sin 2x + 3\sin x = 0$$

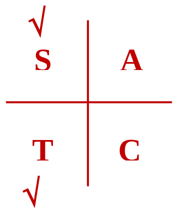
$$2(2\sin x\cos x) + 3\sin x = 0$$

$$4\sin x\cos x + 3\sin x = 0$$

$$\sin x(4\cos x + 3) = 0$$

$\sin x = 0$ or $4\cos x + 3 = 0$

$x = 0°, 180°, \cancel{360}°$ or $\cos x = -\frac{3}{4}$



$x = 180 - 41.41°, 180 + 41.41°$

$x = 138.59°, \cancel{221.41°}$

Always check the range!!

$x = 0°, 138.59° \;\&\; 180°$

13. Solve $3\cos 2\theta + 10\cos\theta = 1$ where $0 \le \theta \le \pi$

$$3\cos 2\theta + 10\cos\theta - 1 = 0$$

$$3(2\cos^2\theta - 1) + 10\cos\theta - 1 = 0$$

Use the cos expansion of cos2x because of the 10cosθ

$$6\cos^2\theta - 3 + 10\cos\theta - 1 = 0$$

$$6\cos^2\theta + 10\cos\theta - 4 = 0$$

$$2(3\cos^2\theta + 5\cos\theta - 2) = 0$$

Consider: $3x^2 + 5x - 2$
$(3x - 1)(x + 2)$

$$2(3\cos\theta - 1)(\cos\theta + 2) = 0$$

$$3\cos\theta - 1 = 0 \quad or \quad \cos\theta + 2 = 0$$

$$\cos\theta = \frac{1}{3} \quad or \quad \cancel{\cos\theta = -2}$$

Not possible!!

$$\theta = 70.53°, 360 - 70.53°$$

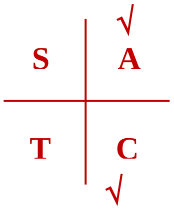
$$\theta = 70.53°, \cancel{289.47°}$$

Not in the range!!

$$\theta = 70.53°$$

$$\theta = 1.23\ radians$$

÷ 180 x π



14. Solve $\cos 2x + 2\sin x = \sin^2 x$ where $0° \le x \le 360°$

$$\cos 2x + 2\sin x - \sin^2 x = 0$$

$$1 - 2\sin^2 x + 2\sin x - \sin^2 x = 0$$

Use the sin expansion of cos2x because of the 2sinx

$$1 + 2\sin x - 3\sin^2 x = 0$$

$$-3\sin^2 x + 2\sin x + 1 = 0$$

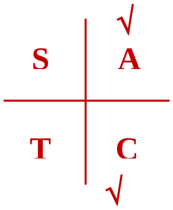
$$-1(3\sin^2 x - 2\sin x - 1) = 0$$

Consider: $3x^2 - 2x - 1$
$(3x + 1)(x - 1)$

$$-1(3\sin x + 1)(\sin x - 1) = 0$$

$$3\sin x + 1 = 0 \quad or \quad \sin x - 1 = 0$$

$$\sin x = -\frac{1}{3} \quad or \quad \sin x = 1$$



$$x = 180 + 19.47°, 360 - 19.47° \qquad x = 90°$$

$$x = 199.47°, 340.53°$$

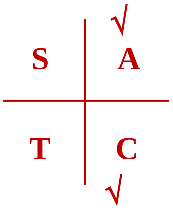
$$x = 90°, 199.47°\ \&\ 340.53°$$

15. Solve $\sin\theta = \sin 2\theta$ where $0 \leq \theta \leq 2\pi$

$$\sin\theta - \sin 2\theta = 0$$

$$\sin\theta - 2\sin\theta\cos\theta = 0$$

$$\sin\theta(1 - 2\cos\theta) = 0$$



$sin\theta = 0$ or $1 - 2\cos\theta = 0$

$\theta = 0°, 180°, 360°$ or $cos\theta = \frac{1}{2}$

$\theta = 60°, 360 - 60°$

$\theta = 0°, 60°, 180°, 300° \;\&\; 360°$

$\theta = 0, \frac{\pi}{3}, \pi, \frac{5\pi}{3} \;\&\; 2\pi$

16. a) Express the function, $f(a) = 6\sin^2 a - \cos a$, in the form $f(a) = p\cos^2 a + q\cos a + r$ and write down the values of p, q and r.

$$f(a) = 6\sin^2 a - \cos a$$

$$f(a) = 6(1 - \cos^2 a) - \cos a$$

$$f(a) = 6 - 6\cos^2 a - \cos a$$

$$f(a) = -6\cos^2 a - \cos a + 6 \qquad p = -6,\; q = -1 \;\&\; r = 6$$

b) Hence, or otherwise, solve $6\sin^2 a - \cos a = 5$ where $0° \leq a \leq 360°$

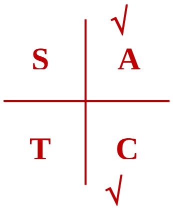
$$6\sin^2 a - \cos a = 5$$

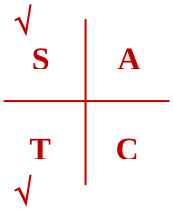
$$-6\cos^2 a - \cos a + 6 = 5$$

$$-6\cos^2 a - \cos a + 1 = 0$$

$$-1(6\cos^2 a + \cos a - 1) = 0$$

$$-1(3\cos a - 1)(2\cos a + 1) = 0$$





$\cos a = \frac{1}{3}$ or $\cos a = -\frac{1}{2}$

$a = 70.53°, 360 - 70.53°$ or $a = 180 - 60°, 180 + 60°$

$a = 70.53°, 289.47°$ or $a = 120°, 240°$

$a = 70.53°, 120°, 240° \;\&\; 289.47°$

Now attempt Exercise 4 from the Compound Angle Formulae booklet

Trig Identities:

- Remember the Trig Identities from Nat 5: $\tan\theta = \frac{\sin\theta}{\cos\theta}$ and $\sin^2 x + \cos^2 x = 1$

  $\sin^2 x = 1 - \cos^2 x$

  $\cos^2 x = 1 - \sin^2 x$

- We will use these Identities along with the 4 Double Angle Formulae to prove Trig Identities:
  - $\sin 2A = 2\sin A\cos A$
  - $\cos 2A = \cos^2 A - \sin^2 A$
  - $\cos 2A = 2\cos^2 A - 1$
  - $\cos 2A = 1 - 2\sin^2 A$
- Always start with the LHS and try to end up with the RHS!!
- Higher Trig Identities are more complex than those you did at Nat 5 as there maybe more than one substitution to make at each step!!

Examples:

17. Show that $\frac{1-\cos 2\theta}{1+\cos 2\theta} = \tan^2\theta$

Need to use $\tan\theta = \frac{\sin\theta}{\cos\theta}$ **in some way!!**

$\frac{1-(1-2\sin^2\theta)}{1+2\cos^2\theta-1}$

Expand cos2*x*

=> $\frac{2\sin^2\theta}{2\cos^2\theta}$

=> $\tan^2\theta$ as required.

18. Show that $3\cos^2 x - \sin^2 x - 1 = 2\cos 2x$.

RHS has no sin so get rid of them from LHS!!

$3\cos^2 x - (1-\cos^2 x) - 1$

$\sin^2 x + \cos^2 x = 1 \rightarrow \sin^2 x = 1 - \cos^2 x$

=> $3\cos^2 x - 1 + \cos^2 x - 1$

=> $4\cos^2 x - 2$

=> $2(2\cos^2 x - 1)$

$2\cos^2 x - 1 = \cos 2x$

=> $2\cos 2x$ as required.

19. Show that $\sin 3x = 3\sin x - 4\sin^3 x$:

$$\sin 3x = \sin(2x + x) = \sin 2x\cos x + \cos 2x\sin x$$

Expand sin2*x* & cos2*x*

$$= (2\sin x\cos x)\cos x + (1 - 2\sin^2 x)\sin x$$

$$= 2\sin x\cos^2 x + \sin x - 2\sin^3 x$$

Rearrange $\cos^2 x + \sin^2 x = 1$ to replace $\cos^2 x$

$$= 2\sin x(1 - \sin^2 x) + \sin x - 2\sin^3 x$$

$$= 2\sin x - 2\sin^3 x + \sin x - 2\sin^3 x$$

$$= 3\sin x - 4\sin^3 x \quad \text{as required}$$

Now attempt Exercise 5 – 7 from the Compound Angle Formulae booklet