H Further Calculus

SPTA Mathematics – Higher Notes

Differentiation – Remember:

- $f(x) = ax^n \implies f'(x) = anx^{n-1}$ where n is a rational number.
- y is the same as $f(x)$ and $\frac{dy}{dx}$ is the same as $f'(x)$.
- The Derivative of a constant is zero.
- Prior to Differentiating a function the following must be true:
  - Until now all brackets needed to be multiplied out. ← **We will see below that this is not always true!!**
  - Roots need to be changed to powers: $\sqrt[m]{x^n} = x^{n/m}$
  - x cannot appear in the denominator of a fraction: $\frac{a}{bx^n} = \frac{a}{b}x^{-n}$
- After Differentiating it is good practice to return the expression to the form the question gave it in.

Chain Rule:

- The CHAIN RULE is a method of Differentiating composite functions.
- We usually concentrate on functions in the form: $y = (f(x))^n \implies \frac{dy}{dx} = n\big(f(x)\big)^{n-1} \times f'(x)$
- A simple way of thinking about the Chain Rule is:

  "*Differentiate outside the bracket multiplied by the derivative of inside the bracket*"

Examples:

1. Differentiate the following with respect to x :

OUTSIDE **INSIDE**

a) $y = (7x^2 - 3x + 5)^5$

$$\frac{dy}{dx} = 5 \times (7x^2 - 3x + 5)^4 \times (14x - 3)$$

$$\frac{dy}{dx} = 5(14x - 3)(7x^2 - 3x + 5)^4$$

b) $f(x) = \frac{1}{(4x+5)^2}, \ x > 0$

$$f(x) = (4x + 5)^{-2} \implies f'(x) = -2 \times (4x + 5)^{-3} \times 4$$

$$\implies f'(x) = \frac{-8}{(4x+5)^3}$$

c) $y = \sqrt[4]{(2x-8)^3}\,,\ x \neq -4$ $y = (2x-8)^{3/4}$

$$\frac{dy}{dx} = \frac{3}{4} \times (2x-8)^{-1/4} \times 2$$

$$\frac{dy}{dx} = \frac{3}{2} \times \frac{1}{(2x-8)^{1/4}}$$

$$\frac{dy}{dx} = \frac{3}{2\sqrt[4]{(2x-8)}}$$

Now attempt Exercise 1 from the Further Calculus booklet.

- Differentiating gives the **<u>Rate of Change</u>** or **<u>GRADIENT</u>** of a curve at the point $x = a$.
- Before substituting in a value for x:
  - Express all negative powers as positive powers: $\frac{a}{b}x^{-n} = \frac{a}{bx^n}$
  - Express all fractional powers as roots: $ax^{n/m} = a\sqrt[m]{x^n}$

<u>Examples:</u>

2. Calculate the Rate of Change of $f(x) = \sqrt{(2x+7)^3}$ at the point $x = -1$:

$f(x) = (2x+7)^{3/2}$

$f'(x) = \frac{3}{2} \times (2x+7)^{1/2} \times 2$

$f'(x) = 3\sqrt{(2x+7)}$

$f'(-1) = 3\sqrt{(2(-1)+7)}$

$= 3\sqrt{5}$

3. Find the equation of the Tangent to the curve $y = 6\sqrt[3]{(2x+2)^2}$ when $x = 3$

$y = 6(2x+2)^{2/3}$ so when $x = 3$

<u>POINT</u>	<u>GRADIENT</u>	<u>EQUATION</u>
$y = 6\sqrt[3]{(2(3)+2)^2}$	$\frac{dy}{dx} = 6 \times \frac{2}{3} \times (2x+2)^{-1/3} \times 2$	$y - b = m(x - a)$
$y = 6\sqrt[3]{8^2}$	$\frac{dy}{dx} = \frac{8}{\sqrt[3]{2x+2}}$	$y - 24 = 4(x - 3)$
$y = 6 \times 4 = 24$	$\frac{dy}{dx} = \frac{8}{2}$ so m = 4	$y - 24 = 4x - 12$
Pt (3 , 24)		$y = 4x + 12$

Now attempt Exercise 2 from the Further Calculus booklet.

Differentiating Trig Functions:

- $y = \sin\theta \implies \frac{dy}{d\theta} = \cos\theta$
- $y = \cos\theta \implies \frac{dy}{d\theta} = -\sin\theta$

Given on your Formulae Sheet

- For these results to be valid the angle **must** be measured in **RADIANS**.
  This is why we usually use θ rather than x

Examples:

4. Differentiate the following with respect to θ:

a) $y = 5\sin\theta$ $\quad \frac{dy}{d\theta} = 5\cos\theta$

b) $f(x) = 3\sin\theta + 2\cos\theta$ $\quad f'(x) = 3\cos\theta - 2\sin\theta$

Now attempt Exercise 3A from the Further Calculus booklet.

Chain Rule for Trig:

- Remember that $y = \sin^n\theta$ actually means $y = (\sin\theta)^n$, similarly for cos so we can use the Chain Rule!
- The Chain Rule can also be used for Differentiating Trig Functions as follows:
  - $y = \sin(a\theta + b) \implies \frac{dy}{dx} = a\cos(a\theta + b)$
  - $y = \cos(a\theta + b) \implies \frac{dy}{dx} = -a\sin(a\theta + b)$

Examples:

5. Differentiate the following with respect to x :

a) $f(x) = 3cos5\theta$ $\quad f(x) = 3\cos(5\theta)$ $\quad f'(x) = -3\sin(5\theta) \times 5$

$f'(x) = -15\sin 5\theta$

b) $f(x) = 2\sin^6\theta$

$f(x) = 2(\sin\theta)^6$

$f'(x) = 6 \times 2(\sin\theta)^5 \times \cos\theta$

$f'(x) = 12\sin^5\theta\cos\theta$

c) $y = \sin\left(7\theta + \frac{\pi}{3}\right)$

$\frac{dy}{dx} = \cos\left(7\theta + \frac{\pi}{3}\right) \times 7$

$\frac{dy}{dx} = 7\cos\left(7\theta + \frac{\pi}{3}\right)$

6. Find $f'(x)$ when $f(x) = \frac{1 - x\cos x}{3x}$

$f(x) = \frac{1 - x\cos x}{3x}$

$f(x) = \frac{1}{3x} - \frac{x\cos x}{3x}$

$f(x) = \frac{1}{3}x^{-1} - \frac{1}{3}\cos x$

$f'(x) = -\frac{1}{3}x^{-2} - \left(-\frac{1}{3}\sin x\right)$

$f'(x) = -\frac{1}{3x^2} + \frac{1}{3}\sin x$

Split into separate fractions before Differentiating, but no need to return to a single fraction unless asked to do so!!

7. For the function $y = 3\sin^2\theta - 2\cos 3\theta$ find $\frac{dy}{d\theta}$ when $\theta = \frac{\pi}{6}$ expressing your answer as a single fraction.

$y = 3(\sin\theta)^2 - 2\cos(3\theta)$

$\frac{dy}{d\theta} = 2 \times 3\sin\theta \times cos\theta - 2 \times -\sin(3\theta) \times 3$

$\frac{dy}{d\theta} = 6\sin\theta cos\theta + 6\sin 3\theta$

$6\sin\frac{\pi}{6} cos\frac{\pi}{6} + 6\sin 3\left(\frac{\pi}{6}\right)$

$= 6 \times \frac{1}{2} \times \frac{\sqrt{3}}{2} + 6\sin\left(\frac{\pi}{2}\right)$

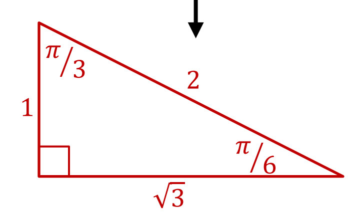
$= 6 \times \frac{1}{2} \times \frac{\sqrt{3}}{2} + 6 \times 1$

$= \frac{3\sqrt{3}}{2} + 6$

$= \frac{3\sqrt{3}}{2} + \frac{6}{1}$

$= \frac{3\sqrt{3}+12}{2}$

Must know the 2 exact value triangles and 3 Trig graphs!



Now attempt Exercise 3B from the Further Calculus booklet.

Integration – Remember:

Must include!

We will see below that this is not always true!!

- $\int ax^n \; dx \; = \frac{ax^{n+1}}{n+1} + c$ where c is the constant of integration.
- The Integral of a constant is x.
- Prior to Integrating a function the following must be true:
  - Until now all brackets needed to be multiplied out.
  - Roots need to be changed to fractional powers: $\sqrt[m]{x^n} = x^{n/m}$
  - x cannot appear in the denominator of a fraction: $\frac{a}{bx^n} = \frac{a}{b}x^{-n}$
- After Integrating it is good practice to return the expression to the form the question gave it in.

A Special Integral:

- A function in the form $y = (ax + b)^n$ can be integrated as follows: $\int (ax+b)^n \, dx \; = \frac{(ax+b)^{n+1}}{a(n+1)} + c$

Examples:

8. Integrate the following:

a) $\int (4x-7)^4 \, dx$

$$= \frac{(4x-7)^5}{4 \times 5} + c$$

$$= \frac{(4x-7)^5}{20} + c$$

b) $\int \frac{6}{\sqrt[3]{3x+2}} dx \;, \; x > 0$

$$\int 6(3x+2)^{-1/3} \, dx$$

$$= \frac{6(3x+2)^{2/3}}{2/3 \times 3}$$

$$= \frac{6\sqrt[3]{(3x+2)^2}}{2}$$

$$= 3\sqrt[3]{(3x+2)^2}$$

Now attempt Exercise 4 Qu 1 & 2 from the Further Calculus booklet.

- Integrating gives the **Area Under the Curve**. These are called **DEFINITE INTEGRALS**.
- Before substituting in a value for x:
  - Express all negative powers as positive powers: $\frac{a}{b}x^{-n} = \frac{a}{bx^n}$
  - Express all fractional powers as roots: $ax^{n/m} = a\sqrt[m]{x^n}$

Examples:

9. $\int_0^4 \sqrt{3x+4}\, dx \ , \ x \geq 0$

$$= \int_0^4 (3x+4)^{1/2}\, dx$$

$$= \left[\frac{(3x+4)^{3/2}}{3/2 \times 3}\right]_0^4$$

$$= \left[\frac{2\sqrt{(3x+4)^3}}{9}\right]_0^4$$

$$= \frac{2\sqrt{(3(4)+4)^3}}{9} - \frac{2\sqrt{(3(0)+4)^3}}{9}$$

$$= \frac{2\sqrt{16^3}}{9} - \frac{2\sqrt{4^3}}{9}$$

$$= \frac{2\times 4^3}{9} - \frac{2\times 2^3}{9}$$

$$= \frac{128}{9} - \frac{16}{9}$$

$$= \frac{112}{9}$$

Units are only required if an Area Qu!!

Now attempt Exercise 4 Qu 3 from the Further Calculus booklet.

Integrating Trig Functions:

- Trig Functions can be Integrated as follows:
  - $\int \sin(ax+b)\, dx = -\frac{1}{a}\cos(ax+b) + c$
  - $\int \cos(ax+b)\, dx = \frac{1}{a}\sin(ax+b) + c$

Given on your Formulae Sheet

- Again, for these results to be valid the angle **<u>must</u>** be measured in **<u>RADIANS</u>**.

Examples:

10. Integrate the following:

a) $\int \sin(3x-1)\, dx$

$$= -\frac{1}{3}\cos(3x-1) + c$$

b) $\int 3\cos\left(\frac{3}{4}x+2\right) dx$

$$= 3\sin\left(\frac{3}{4}x+2\right) \div \frac{3}{4} + c$$

$$= 3 \times \frac{4}{3}\sin\left(\frac{3}{4}x+2\right) + c$$

$$= 4\sin\left(\frac{3}{4}x+2\right) + c$$

c) $\int 5\cos 2x + \sin\left(x - \sqrt{3}\right)\,dx$ $\quad = \frac{1}{2} \times 5\sin 2x - \cos\left(x - \sqrt{3}\right) + c$

$= \frac{5}{2}\sin 2x - \cos\left(x - \sqrt{3}\right) + c$

Now attempt Exercise 5 Qu 1 & 2 from the Further Calculus booklet.

- We cannot Integrate Trig functions in the forms below without first using an **ADDITION FORMULAE** expansion:
  - $\int sin^2\theta\, d\theta \;\; or \;\; \int cos^2\theta\, d\theta$
  - $\int \sin\theta \cos\theta\, d\theta$ or a similar **PRODUCT** of Trig expressions

- The Addition Formulae expansions are:
  - $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$
  - $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$
  - $\sin 2A = 2 \sin A \cos A$
  - $\cos 2A = \cos^2 A - \sin^2 A$
  - $\cos 2A = 2\cos^2 A - 1$
  - $\cos 2A = 1 - 2\sin^2 A$

Examples:

11. a) $\int 5 \cos^2 x\, dx$

$\cos 2x = 2\cos^2 x - 1$

$\cos 2x + 1 = 2\cos^2 x$

$\cos^2 x = ½\cos 2x + ½$

$= \int 5\left(\frac{1}{2}\cos 2x + \frac{1}{2}\right) dx$

$= \int \frac{5}{2}\cos 2x + \frac{5}{2}\, dx$

$= \frac{5}{2} \times \frac{1}{2}\sin 2x + \frac{5}{2}x + c$

$= \frac{5}{4}\sin 2x + \frac{5}{2}x + c$

b) $\int \sin x \cos x \, dx$

$\sin 2x = 2\sin x\cos x$

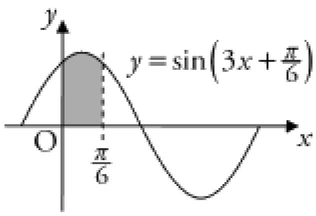
$\frac{1}{2}\sin 2x = \sin x\cos x$

$= \int \frac{1}{2}\sin 2x \, dx$

$= \frac{1}{2} \times -\frac{1}{2}\cos 2x + c$

$= -\frac{1}{4}\cos 2x + c$

12. Find this shaded area:



$\int_0^{\frac{\pi}{6}} \sin\left(3x + \frac{\pi}{6}\right) dx$

$= \left[-\frac{1}{3}\cos\left(3x + \frac{\pi}{6}\right)\right]_0^{\frac{\pi}{6}}$

$= -\frac{1}{3}\cos\left(3\left(\frac{\pi}{6}\right) + \frac{\pi}{6}\right) - \left(-\frac{1}{3}\cos\left(3(0) + \frac{\pi}{6}\right)\right)$

$= -\frac{1}{3}\cos\left(\frac{3\pi}{6} + \frac{\pi}{6}\right) - \left(-\frac{1}{3}\cos\left(\frac{\pi}{6}\right)\right)$

$= -\frac{1}{3}\cos\left(\frac{4\pi}{6}\right) + \frac{1}{3}\cos\left(\frac{\pi}{6}\right)$

$= -\frac{1}{3} \times \left(-\frac{1}{2}\right) + \frac{1}{3} \times \left(\frac{\sqrt{3}}{2}\right)$

$= \frac{1}{6} + \frac{\sqrt{3}}{6}$

$= \frac{1+\sqrt{3}}{6}$ so the Area is $\frac{1+\sqrt{3}}{6}$ units^2

Units are required as it's an Area Qu!!

Now attempt Exercise 5 Qu 3 & 4 from the Further Calculus booklet.

Differential Equations – Remember:

- A Differential Equation is any equation which contains $\frac{dy}{dx}$ or $f'(x)$ e.g. $\frac{dy}{dx} = 4x - 5$
- In Higher we only consider first order differential equations.
- To solve a Differential Equation we Integrate to find the General Solution which involves $+c$.
- With more information, such as a point we can find the Particular Solution not involving c.

Examples:

13. Find the Particular Solution of the Differential Equation $\frac{dy}{dx} = 3\sin 3x$ given that $y = 2$ when $x = \frac{\pi}{3}$

$y = \int 3\sin 3x$ $\quad$ $y = 3 \times -\frac{1}{3}\cos 3x + c$

$y = -\cos 3x + c$ ← **General Solution**

When $= \frac{\pi}{3}, y = 2$ $\quad$ $2 = -\cos 3\left(\frac{\pi}{3}\right) + c$

$2 = -\cos(\pi) + c$

$2 = -(-1) + c$

$2 = 1 + c$

$c = 1$

$y = -\cos 3x + 1$ ← **Particular Solution**

Now attempt Exercise 6 – 8 from the Further Calculus booklet.