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Higher Mathematics

Vectors

Contents

CfE Edition

Vectors

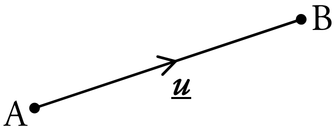
1 Vectors and Scalars EF

A **scalar** is a quantity with magnitude (size) only – for example, an amount of money or a length of time.

Sometimes size alone is not enough to describe a quantity – for example, directions to the nearest shop. For this we need to know a magnitude (i.e. how far), *and* a direction.

Quantities with magnitude and direction are called **vectors**.

A vector is named either by using the letters at the end of a directed line segment (e.g. $\overrightarrow{AB}$ represents a vector starting at point A and ending at point B) or by using a bold letter (e.g. $\boldsymbol{u}$). You will see bold letters used in printed text, but in handwriting you should just underline the letter.



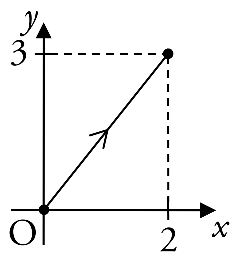
Throughout these notes, we will show vectors in bold as well us underlining them (e.g. $\underline{\boldsymbol{u}}$).

2 Components EF

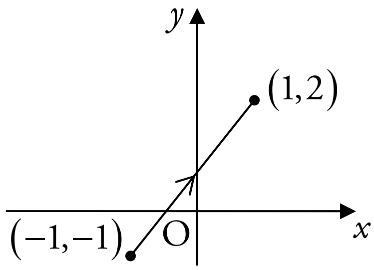
A vector may be represented by its **components**, which we write in a column. For example,

$$\begin{pmatrix} 2 \\ 3 \end{pmatrix} \text{ is a vector in two dimensions.}$$

In this case, the first component is 2 and this tells us to move 2 units in the x-direction. The second component tells us to move 3 units in the y-direction. So if the vector starts at the origin, it will look like:



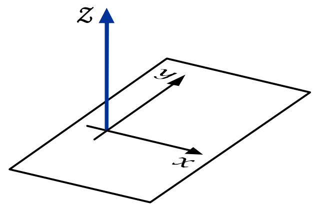
Note that we write the components in a column to avoid confusing them with coordinates. The following diagram also shows the vector $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$, but in this case it does not start at the origin.



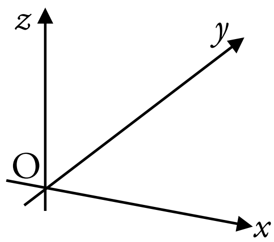
Vectors in Three Dimensions

In a vector with three components, the first two tell us ho many units to move in the *x*- and *y*-directions, as before. The third component specifies how far to move in the *z*-direction.

When looking at a pair of (x, y) -axes, the *z*-axis points out of the page from the origin.



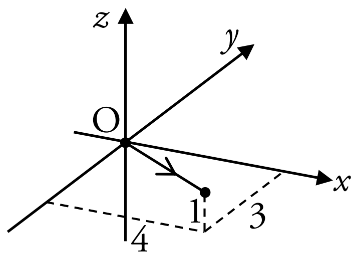
A set of 3D axes can be drawn on a page as shown to the right.



For example,

$$\begin{pmatrix} 4 \\ -3 \\ 1 \end{pmatrix}$$

is a vector in three dimensions. This vector is shown in the diagram, starting from the origin.



Zero Vectors

Any vector with all components zero is called a zero vector and can be written as $\underline{\mathbf{0}}$, e.g. $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \underline{\mathbf{0}}$.

3 Magnitude EF

The **magnitude** (or length) of a vector $\underline{u}$ is written as $|\underline{u}|$. It can be calculated as follows.

$$\text{If} \quad \boldsymbol{u} = \begin{pmatrix} a \\ b \end{pmatrix} \quad \text{then} \quad |\underline{u}| = \sqrt{a^2 + b^2}$$

$$\text{If} \quad \boldsymbol{u} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad \text{then} \quad |\underline{u}| = \sqrt{a^2 + b^2 + c^2}$$

EXAMPLES

1. Given $\underline{u} = \begin{pmatrix} 5 \\ -12 \end{pmatrix}$, find $|\underline{u}|$.

$$\begin{aligned} |\underline{u}| &= \sqrt{(5)^2 + (-12)^2} \\ &= \sqrt{169} \\ &= 13 \text{ units.} \end{aligned}$$

2. Find the length of $\underline{a} = \begin{pmatrix} -\sqrt{5} \\ 6 \\ 3 \end{pmatrix}$.

$$\begin{aligned} |\underline{a}| &= \sqrt{\left(-\sqrt{5}\right)^2 + 6^2 + 3^2} \\ &= \sqrt{50} \\ &= 5\sqrt{2} \text{ units.} \end{aligned}$$

Unit Vectors

Any vector with a magnitude of one is called a **unit vector**. For example:

$$\text{if } \underline{u} = \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{\sqrt{3}}{2} \end{pmatrix} \text{ then } \begin{aligned}[t] |\underline{u}| &= \sqrt{\left(\tfrac{1}{2}\right)^2 + 0^2 + \left(\tfrac{\sqrt{3}}{2}\right)^2} \\ &= \sqrt{\tfrac{4}{4}} \\ &= 1 \text{ unit.} \end{aligned}$$

So $\underline{u}$ is a unit vector.

Distance in Three Dimensions

The distance between the points A and B is $d_{\text{AB}} = \left|\overrightarrow{\text{AB}}\right|$ units.

For example, given $\overrightarrow{\text{AB}} = \begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix}$, we find $d_{\text{AB}} = \sqrt{(-1)^2 + 2^2 + 5^2} = \sqrt{30}$.

In fact, there is a three-dimensional version of the distance formula.

The distance d between the points (x_1, y_1, z_1) and (x_1, y_1, z_1) is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \text{ units.}$$

EXAMPLE

Find the distance between the points $(-1,4,1)$ and $(0,5,-7)$.

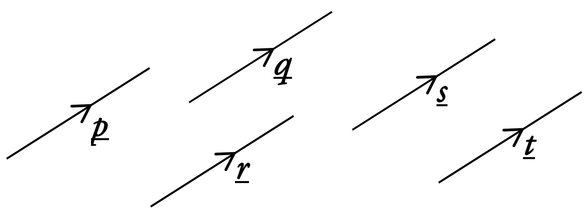
$$\begin{aligned} \text{The distance is } & \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \\ &= \sqrt{(0-(-1))^2 + (5-4)^2 + (-7-1)^2} \\ &= \sqrt{1^2 + 1^2 + (-8)^2} \\ &= \sqrt{1+1+64} \\ &= \sqrt{66} \text{ units.} \end{aligned}$$

4 Equal Vectors

EF

Vectors with the same magnitude and direction are equal.

For example, all the vectors shown to the right are equal.



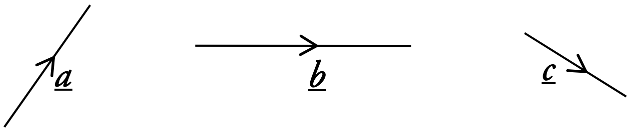
If vectors are equal to each other, then all of their components are equal, i.e.

$$\text{if } \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} d \\ e \\ f \end{pmatrix} \text{ then } a = d,\ b = e \text{ and } c = f.$$

Conversely, two vectors are only equal if all of their components are equal.

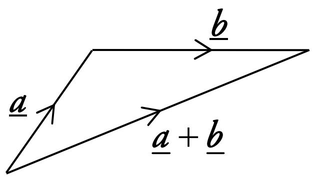
5 Addition and Subtraction of Vectors EF

Consider the following vectors:



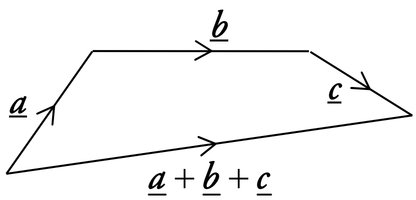
Addition

We can construct $\underline{a}+\underline{b}$ as follows:



$\underline{a}+\underline{b}$ means $\underline{a}$ followed by $\underline{b}$.

Similarly, we can construct $\underline{a}+\underline{b}+\underline{c}$ as follows:

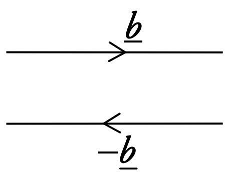


$\underline{a}+\underline{b}+\underline{c}$ means $\underline{a}$ followed by $\underline{b}$ followed by $\underline{c}$.

To add vectors, we position them nose-to-tail. Then the sum of the vectors is the vector between the first tail and the last nose.

Subtraction

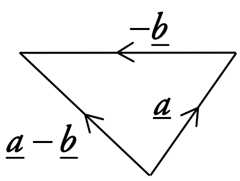
Now consider $\underline{a}-\underline{b}$. This can be written as $\underline{a}+(-\underline{b})$, so if we first find $-\underline{b}$, we can use vector addition to obtain $\underline{a}-\underline{b}$.



$-\underline{b}$ is just $\underline{b}$ but in the opposite direction.

$-\underline{b}$ and $\underline{b}$ have the same magnitude, i.e. $|\underline{b}|=|-\underline{b}|$.

Therefore we can construct $\underline{a}-\underline{b}$ as follows:



$\underline{a}-\underline{b}$ means $\underline{a}$ followed by $-\underline{b}$.

Using Components

If we have the components of vectors, then things become much simpler.

The following rules can be used for addition and subtraction.

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} + \begin{pmatrix} d \\ e \\ f \end{pmatrix} = \begin{pmatrix} a+d \\ b+e \\ c+f \end{pmatrix}$$

add the components

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} - \begin{pmatrix} d \\ e \\ f \end{pmatrix} = \begin{pmatrix} a-d \\ b-e \\ c-f \end{pmatrix}$$

subtract the components

EXAMPLES

1. Given $\underline{u} = \begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix}$ and $\underline{v} = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$, calculate $\underline{u} + \underline{v}$ and $\underline{u} - \underline{v}$.

$$\underline{u} + \underline{v} = \begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix} + \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 7 \\ 2 \end{pmatrix}$$

$$\underline{u} - \underline{v} = \begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix} - \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}.$$

2. Given $\underline{p} = \begin{pmatrix} 4 \\ \frac{3}{2} \\ 3 \end{pmatrix}$ and $\underline{q} = \begin{pmatrix} -1 \\ 3 \\ -\frac{6}{5} \end{pmatrix}$, calculate $\underline{p} - \underline{q}$ and $\underline{q} + \underline{p}$.

$$\underline{p} - \underline{q} = \begin{pmatrix} 4 \\ \frac{3}{2} \\ 3 \end{pmatrix} - \begin{pmatrix} -1 \\ 3 \\ -\frac{6}{5} \end{pmatrix} = \begin{pmatrix} 5 \\ -\frac{3}{2} \\ \frac{21}{5} \end{pmatrix}$$

$$\underline{q} + \underline{p} = \begin{pmatrix} -1 \\ 3 \\ -\frac{6}{5} \end{pmatrix} + \begin{pmatrix} 4 \\ \frac{3}{2} \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ \frac{9}{2} \\ \frac{9}{5} \end{pmatrix}.$$

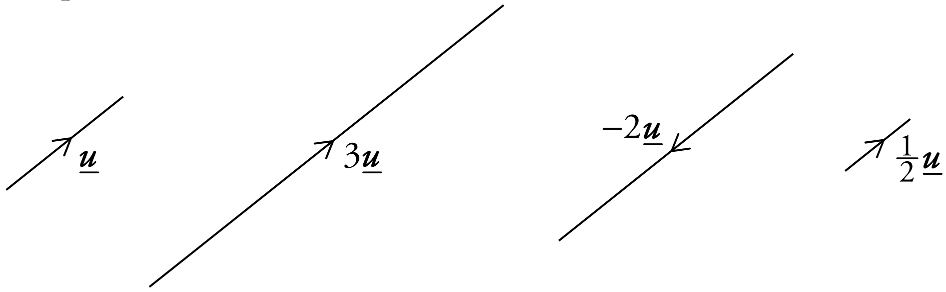
6 Multiplication by a Scalar

EF

A vector $\underline{u}$ which is multiplied by a scalar $k > 0$ will give the result $k\underline{u}$. This vector will be k times as long, i.e. the magnitude will be $k|\underline{u}|$.

Note that if $k < 0$ this means that the vector $k\underline{u}$ will be in the opposite direction to $\underline{u}$.

For example:



$$\text{If} \quad \underline{u} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad \text{then} \quad k\underline{u} = \begin{pmatrix} ka \\ kb \\ kc \end{pmatrix}.$$

Each component is multiplied by the scalar.

EXAMPLES

1. Given $\underline{v} = \begin{pmatrix} 1 \\ 5 \\ -3 \end{pmatrix}$, find $3\underline{v}$.

$$3\underline{v} = 3\begin{pmatrix} 1 \\ 5 \\ -3 \end{pmatrix} = \begin{pmatrix} 3 \\ 15 \\ -9 \end{pmatrix}.$$

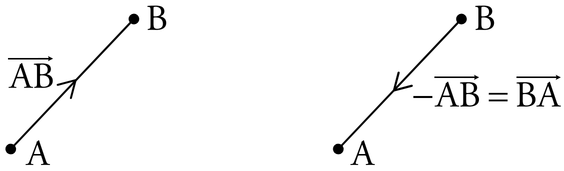
2. Given $\underline{r} = \begin{pmatrix} -6 \\ 3 \\ 1 \end{pmatrix}$, find $-4\underline{r}$.

$$-4\underline{r} = -4\begin{pmatrix} -6 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 24 \\ -12 \\ -4 \end{pmatrix}.$$

Negative Vectors

The negative of a vector is the vector multiplied by -1.

If we write a vector as a directed line segment $\overrightarrow{AB}$, then $-\overrightarrow{AB} = \overrightarrow{BA}$:



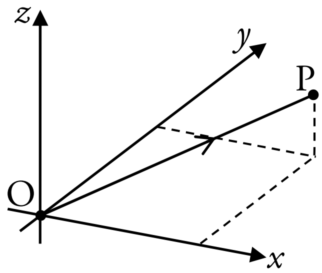
7 Position Vectors

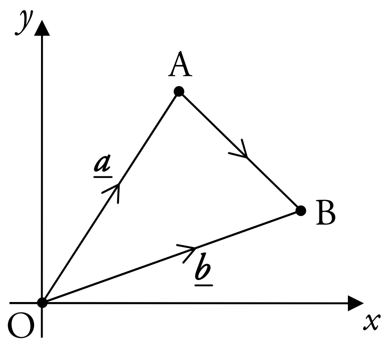
EF

$\overrightarrow{OA}$ is called a **position vector** of point A relative to the origin O, and is written as $\underset{\sim}{a}$.

$\overrightarrow{OB}$ is called the position vector of point B, written $\underset{\sim}{b}$.

Given $P(x, y, z)$, the position vector $\overrightarrow{OP}$ or $\underset{\sim}{p}$ has components $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$.





To move from point A to point B we can move back along the vector $\underset{\sim}{a}$ to the origin, and along vector $\underset{\sim}{b}$ to point B.

$$\begin{aligned} \overrightarrow{AB} &= \overrightarrow{AO} + \overrightarrow{OB} \\ &= -\overrightarrow{OA} + \overrightarrow{OB} \\ &= -\underset{\sim}{a} + \underset{\sim}{b} \\ &= \underset{\sim}{b} - \underset{\sim}{a} \end{aligned}$$

For the vector joining any two points P and Q, $\overrightarrow{PQ} = \underset{\sim}{q} - \underset{\sim}{p}$.

EXAMPLE

R is the point $(2, -2, 3)$ and S is the point $(4, 6, -1)$. Find $\overrightarrow{RS}$.

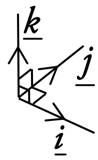
From the coordinates, $\underline{r} = \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}$ and $\underline{s} = \begin{pmatrix} 4 \\ 6 \\ -1 \end{pmatrix}$.

$$\begin{aligned} \overrightarrow{RS} &= \underline{s} - \underline{r} \\ &= \begin{pmatrix} 4 \\ 6 \\ -1 \end{pmatrix} - \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ 8 \\ -4 \end{pmatrix}. \end{aligned}$$

8 Basis Vectors

EF

A vector may also be defined in terms of the basis vectors $\underline{i}$, $\underline{j}$ and $\underline{k}$. These are three mutually perpendicular unit vectors (i.e. they are perpendicular to each other).



These basis vectors can be written in component form as

$$\underline{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \ \underline{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{ and } \underline{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Any vector can be written in **basis form** using $\underline{i}$, $\underline{j}$ and $\underline{k}$. For example:

$$\begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - 3\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 6\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 2\underline{i} - 3\underline{j} + 6\underline{k}.$$

There is no need for the working above if the following is used:

$$a\underline{i} + b\underline{j} + c\underline{k} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

9 Collinearity

EF

In Straight Lines, we learned that points are collinear if they lie on the same straight line.

The points A, B and C in 3D space are collinear if $\overrightarrow{AB}$ is parallel to $\overrightarrow{BC}$, with B a common point.

Note that we *cannot* find gradients in three dimensions – instead we use the following.

Non-zero vectors are **parallel** if they are scalar multiples of the same vector.

For example:

If $\underset{\sim}{u} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}$ and $\underset{\sim}{v} = \begin{pmatrix} 6 \\ 3 \\ 12 \end{pmatrix} = 3\begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} = 3\underset{\sim}{u}$ then $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are parallel.

If $\underset{\sim}{p} = \begin{pmatrix} 15 \\ 9 \\ -6 \end{pmatrix} = 3\begin{pmatrix} 5 \\ 3 \\ -2 \end{pmatrix}$ and $\underset{\sim}{q} = \begin{pmatrix} 20 \\ 12 \\ -8 \end{pmatrix} = 4\begin{pmatrix} 5 \\ 3 \\ -2 \end{pmatrix}$ then $\underset{\sim}{p}$ and $\underset{\sim}{q}$ are parallel.

EXAMPLE

A is the point $(1, -2, 5)$, $B(8, -5, 9)$ and $C(22, -11, 17)$.

Show that A, B and C are collinear.

$$\begin{aligned} \overrightarrow{AB} &= \underset{\sim}{b} - \underset{\sim}{a} \\ &= \begin{pmatrix} 8 \\ -5 \\ 9 \end{pmatrix} - \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix} \\ &= \begin{pmatrix} 7 \\ -3 \\ 4 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \overrightarrow{BC} &= \underset{\sim}{c} - \underset{\sim}{b} \\ &= \begin{pmatrix} 22 \\ -11 \\ 17 \end{pmatrix} - \begin{pmatrix} 8 \\ -5 \\ 9 \end{pmatrix} \\ &= \begin{pmatrix} 14 \\ -6 \\ 8 \end{pmatrix} \\ &= 2\begin{pmatrix} 7 \\ -3 \\ 4 \end{pmatrix}. \end{aligned}$$

$\overrightarrow{BC} = 2\overrightarrow{AB}$, so $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are parallel – and since B is a common point, A, B and C are collinear.

10 Dividing Lines in a Ratio

EF

There is a simple process for finding the coordinates of a point which divides a line segment in a given ratio.

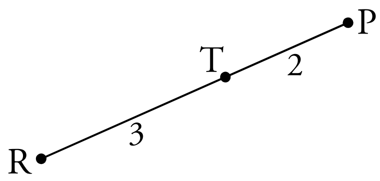
EXAMPLE

1. P is the point $(-2, 4, -1)$ and R is the point $(8, -1, 19)$.

   The point T divides PR in the ratio $2:3$. Find the coordinates of T.

Step 1

Make a sketch of the line, showing the ratio in which the point divides the line segment.



Step 2

Using the sketch, equate the ratio of the two lines with the given ratio.

$$\frac{\text{PT}}{\text{TR}} = \frac{2}{3}$$

Step 3

Cross multiply, then change directed line segments to position vectors.

$$3\overrightarrow{\text{PT}} = 2\overrightarrow{\text{TR}}$$

$$3(\underline{t} - \underline{p}) = 2(\underline{r} - \underline{t})$$

Step 4

Rearrange to give the position vector of the unknown point.

$$3\underline{t} - 3\underline{p} = 2\underline{r} - 2\underline{t}$$

$$3\underline{t} + 2\underline{t} = 2\underline{r} + 3\underline{p}$$

$$5\underline{t} = 2\begin{pmatrix} 8 \\ -1 \\ 19 \end{pmatrix} + 3\begin{pmatrix} -2 \\ 4 \\ -1 \end{pmatrix}$$

$$5\underline{t} = \begin{pmatrix} 16 \\ -2 \\ 38 \end{pmatrix} + \begin{pmatrix} -6 \\ 12 \\ -3 \end{pmatrix}$$

$$5\underline{t} = \begin{pmatrix} 10 \\ 10 \\ 35 \end{pmatrix}$$

$$\underline{t} = \begin{pmatrix} 2 \\ 2 \\ 7 \end{pmatrix}$$

Step 5

From the position vector, state the coordinates of the unknown point.

So T is the point $(2, 2, 7)$.

Using the Section Formula

The previous method can be condensed into a formula as shown below.

If the point P divides the line AB in the ratio $m:n$, then:

$$\underset{\sim}{p} = \frac{n\underset{\sim}{a} + m\underset{\sim}{b}}{n+m},$$

where is $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{p}$ are the position vectors of A, B and P respectively.

It is not necessary to know this, since the approach explained above will always work.

EXAMPLE

2. P is the point $(-2, 4, -1)$ and R is the point $(8, -1, 19)$.

   The point T divides PR in the ratio $2:3$. Find the coordinates of T.

The ratio is $2:3$, so let $m = 2$ and $n = 3$, then:

$$\begin{aligned}
\underset{\sim}{t} &= \frac{n\underset{\sim}{p} + m\underset{\sim}{r}}{n+m} \\
&= \frac{3\underset{\sim}{p} + 2\underset{\sim}{r}}{5} \\
&= \begin{pmatrix} \frac{1}{5}[3(-2)+2(8)] \\ \frac{1}{5}[3(4)+2(-1)] \\ \frac{1}{5}[3(-1)+2(19)] \end{pmatrix} \\
&= \begin{pmatrix} 2 \\ 2 \\ 7 \end{pmatrix}
\end{aligned}$$

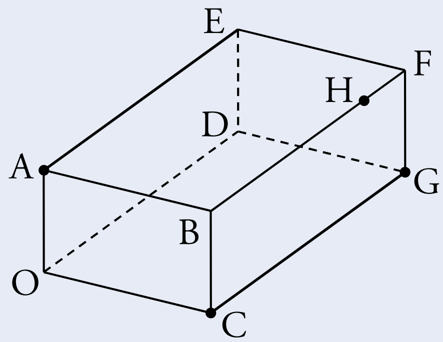
Note
If you are confident with arithmetic, this step can be done mentally.

So T is the point $(2, 2, 7)$.

Further Examples

EXAMPLES

3. The cuboid OABCDEFG is shown in the diagram.



The point A has coordinates $(0,0,5)$, $C(8,0,0)$ and $G(8,12,0)$. The point H divides BF in the ratio $4:1$. Find the coordinates of H.

From the diagram:

$$\begin{aligned}\overrightarrow{OH} &= \overrightarrow{OA} + \overrightarrow{AB} + \tfrac{4}{5}\overrightarrow{BF} \\ &= \overrightarrow{OA} + \overrightarrow{OC} + \tfrac{4}{5}\overrightarrow{CG}\end{aligned}$$

$$\begin{aligned}\underset{\sim}{h} &= \underset{\sim}{a} + \underset{\sim}{c} + \tfrac{4}{5}\left(\underset{\sim}{g} - \underset{\sim}{c}\right) \\ &= \underset{\sim}{a} + \underset{\sim}{c} + \tfrac{4}{5}\underset{\sim}{g} - \tfrac{4}{5}\underset{\sim}{c} \\ &= \underset{\sim}{a} + \tfrac{1}{5}\underset{\sim}{c} + \tfrac{4}{5}\underset{\sim}{g} \\ &= \begin{pmatrix}0\\0\\5\end{pmatrix} + \tfrac{1}{5}\begin{pmatrix}8\\0\\0\end{pmatrix} + \tfrac{4}{5}\begin{pmatrix}8\\12\\5\end{pmatrix} \\ &= \begin{pmatrix}8\\ \frac{48}{5}\\5\end{pmatrix}.\end{aligned}$$

So H has coordinates $\left(8, \tfrac{48}{5}, 5\right)$.

Note

$\frac{BH}{BF} = \frac{4}{5}$, so $\overrightarrow{BH} = \tfrac{4}{5}\overrightarrow{BF}$.

4. The points $P(6,1,-3)$, $Q(8,-3,1)$ and $R(9,-5,3)$ are collinear. Find the ratio in which Q divides PR.

Since the points are collinear $\overrightarrow{PQ} = k\overrightarrow{QR}$ for some k. Working with the first components:

$$\begin{aligned}8 - 6 &= k(9-8) \\ k &= 2.\end{aligned}$$

Therefore $\overrightarrow{PQ} = 2\overrightarrow{QR}$ so Q divides PR in the ratio $2:1$.

Note

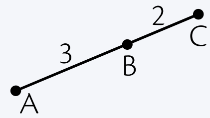
The ratio is $2:1$ since $\frac{PQ}{QR} = \frac{2}{1}$.

5. The points $A(7,-4,-4)$, $B(13,5,-7)$ and C are collinear. Given that B divides AC in the ratio $3:2$, find the coordinates of C.

$$\overrightarrow{AB} = \tfrac{3}{5}\overrightarrow{AC}$$
$$\underline{b} - \underline{a} = \tfrac{3}{5}(\underline{c} - \underline{a})$$
$$\underline{b} - \underline{a} = \tfrac{3}{5}\underline{c} - \tfrac{3}{5}\underline{a}$$
$$\tfrac{3}{5}\underline{c} = \underline{b} - \tfrac{2}{5}\underline{a}$$
$$\underline{c} = \tfrac{5}{3}\underline{b} - \tfrac{2}{3}\underline{a}$$
$$= \tfrac{5}{3}\begin{pmatrix}13\\5\\-7\end{pmatrix} - \tfrac{2}{3}\begin{pmatrix}7\\-4\\-4\end{pmatrix}$$
$$= \begin{pmatrix}17\\11\\-9\end{pmatrix}.$$

So C has coordinates $(17,11,-9)$.

Note
A sketch may help you to see this:



11 The Scalar Product EF

So far we have added and subtracted vectors and multiplied a vector by a scalar. Now we will consider the scalar product, which is a form of vector multiplication.

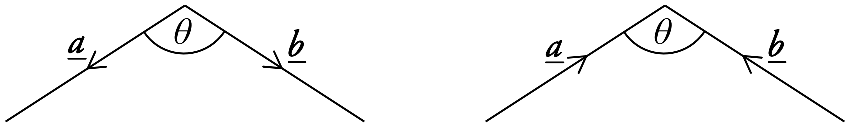
The **scalar product** is denoted by $\underline{a}.\underline{b}$ (sometimes it is called the dot product) and can be calculated using the formula:

$$\underline{a}.\underline{b} = |\underline{a}||\underline{b}|\cos\theta,$$

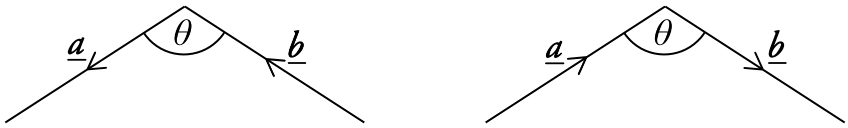
where θ is the angle between the two vectors $\underline{a}$ and $\underline{b}$.

This formula is given in the exam.

The definition above assumes that the vectors $\underline{a}$ and $\underline{b}$ are positioned so that they both point away from the angle, or both point into the angle.



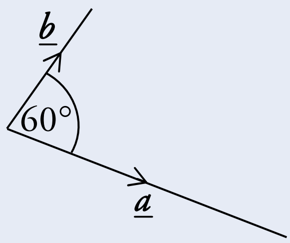
However, if one vector is pointing away from the angle, while the other points into the angle,



we find that $\underline{a}.\underline{b} = -|\underline{a}||\underline{b}|\cos\theta$.

EXAMPLES

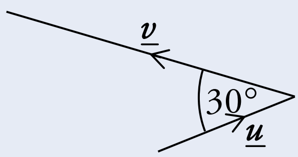
1. Two vectors, $\underline{a}$ and $\underline{b}$ have magnitudes 7 and 3 units respectively and are at an angle of $60°$ to each other as shown below.



What is the value of $\underline{a}.\underline{b}$?

$$\begin{aligned}\underline{a}.\underline{b} &= |\underline{a}||\underline{b}|\cos\theta \\ &= 7\times 3\times\cos 60° \\ &= 21\times\tfrac{1}{2} \\ &= \tfrac{21}{2}.\end{aligned}$$

2. The vector $\underline{u}$ has magnitude k and $\underline{v}$ is twice as long as $\underline{u}$. The angle between $\underline{u}$ and $\underline{v}$ is $30°$, as shown below.



Find an expression for $\underline{u}.\underline{v}$ in terms of k.

$$\begin{aligned}\underline{u}.\underline{v} &= -|\underline{u}||\underline{v}|\cos\theta \\ &= -k\times 2k\times\cos 30° \\ &= -2k^2\times\tfrac{\sqrt{3}}{2} \\ &= -\sqrt{3}k^2.\end{aligned}$$

Remember
When one vector points in and one points out, $\underline{u}.\underline{v} = -|\underline{u}||\underline{v}|\cos\theta$.

The Component Form of the Scalar Product

The scalar product can also be calculated as follows:

$$\underset{\sim}{a}.\underset{\sim}{b} = a_1b_1 + a_2b_2 + a_3b_3 \quad \text{where } \underset{\sim}{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \text{ and } \underset{\sim}{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

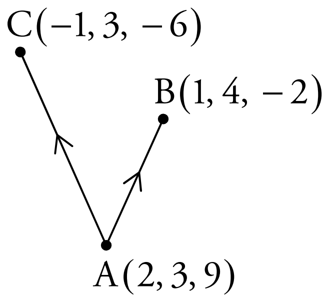
This is given in the exam.

EXAMPLES

3. Find $\underset{\sim}{p}.\underset{\sim}{q}$, given that $\underset{\sim}{p} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$ and $\underset{\sim}{q} = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$.

$$\begin{aligned} \underset{\sim}{p}.\underset{\sim}{q} &= p_1q_1 + p_2q_2 + p_3q_3 \\ &= (1\times 2) + (2\times 2) + ((-3)\times 3) \\ &= 2 + 4 - 9 \\ &= -3 \end{aligned}$$

4. If A is the point $(2, 3, 9)$, $B(1, 4, -2)$ and $C(-1, 3, -6)$, calculate $\overrightarrow{AB}.\overrightarrow{AC}$.



We need to use the position vectors of the points:

$$\overrightarrow{AB} = \underset{\sim}{b} - \underset{\sim}{a} = \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ 9 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ -11 \end{pmatrix}$$

$$\overrightarrow{AC} = \underset{\sim}{c} - \underset{\sim}{a} = \begin{pmatrix} -1 \\ 3 \\ -6 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ 9 \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \\ -15 \end{pmatrix}.$$

$$\begin{aligned} \overrightarrow{AB}.\overrightarrow{AC} &= ((-1)\times(-3)) + (1\times 0) + ((-11)\times(-15)) \\ &= 3 + 0 + 165 \\ &= 168. \end{aligned}$$

12 The Angle Between Vectors

EF

The formulae for the scalar product can be rearranged to give the following equations, both of which can be used to calculate θ, the angle between two vectors.

$$\cos\theta = \frac{\underset{\sim}{a}.\underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|} \qquad \text{or} \qquad \cos\theta = \frac{a_1b_1 + a_2b_2 + a_3b_3}{|\underset{\sim}{a}||\underset{\sim}{b}|}.$$

Look back to the formulae for finding the scalar product, given on the previous pages. Notice that the first equation is simply a rearranged form of the one which can be used to find the scalar product. Also notice that the second simply replaces $\underset{\sim}{a}.\underset{\sim}{b}$ with the component form of the scalar product.

These formulae are *not* given in the exam but can both be easily derived from the formulae on the previous pages (which *are* given in the exam).

EXAMPLES

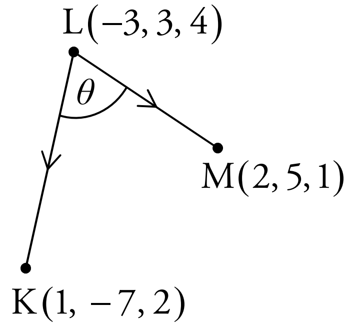
1. Calculate the angle θ between vectors $\underset{\sim}{p} = 3\underset{\sim}{i} + 4\underset{\sim}{j} - 2\underset{\sim}{k}$ and $\underset{\sim}{q} = 4\underset{\sim}{i} + \underset{\sim}{j} + 3\underset{\sim}{k}$.

$$\underset{\sim}{p} = \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix} \text{ and } \underset{\sim}{q} = \begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix}$$

$$\begin{aligned}
\cos\theta &= \frac{p_1q_1 + p_2q_2 + p_3q_3}{|\underset{\sim}{p}||\underset{\sim}{q}|} \\
&= \frac{(3\times 4)+(4\times 1)+((-2)\times 3)}{\sqrt{3^2+4^2+(-2)^2}\sqrt{4^2+1^2+3^2}} \\
&= \frac{10}{\sqrt{29}\sqrt{26}} \\
\theta &= \cos^{-1}\left(\frac{10}{\sqrt{29}\sqrt{26}}\right) \\
&= 68{\cdot}6^\circ \text{ (to 1 d.p.)} \quad \text{(or } 1{\cdot}198 \text{ radians (to 3 d.p.))}
\end{aligned}$$

2. K is the point $(1, -7, 2)$, $L(-3, 3, 4)$ and $M(2, 5, 1)$. Find $K\hat{L}M$.

Start with a sketch:



Now find the vectors pointing away from the angle:

$$\overrightarrow{LK} = \underline{k} - \underline{l} = \begin{pmatrix} 1 \\ -7 \\ 2 \end{pmatrix} - \begin{pmatrix} -3 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 4 \\ -10 \\ -2 \end{pmatrix},$$

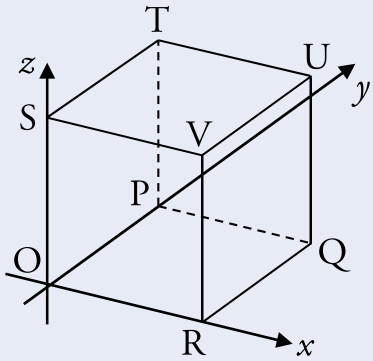
$$\overrightarrow{LM} = \underline{m} - \underline{l} = \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix} - \begin{pmatrix} -3 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ -3 \end{pmatrix}.$$

Use the scalar product to find the angle:

$$\begin{aligned} \cos\theta &= \frac{\overrightarrow{LK}.\overrightarrow{LM}}{\left|\overrightarrow{LK}\right|\left|\overrightarrow{LM}\right|} \\ &= \frac{(4\times 5)+(-10\times 2)+(-2\times(-3))}{\sqrt{4^2+(-10)^2+(-2)^2}\sqrt{5^2+2^2+(-3)^2}} \\ &= \frac{6}{\sqrt{120}\sqrt{38}} \end{aligned}$$

$$\begin{aligned} \theta &= \cos^{-1}\left(\frac{6}{\sqrt{120}\sqrt{38}}\right) \\ &= 84{\cdot}9^\circ \text{ (to 1 d.p.)} \quad \text{(or } 1{\cdot}48 \text{ radians (to 3 d.p.))} \end{aligned}$$

3. The diagram below shows the cube OPQRSTUV.



The point R has coordinates $(4,0,0)$.

(a) Write down the coordinates of T and U.

(b) Find the components of $\overrightarrow{RT}$ and $\overrightarrow{RU}$.

(c) Calculate the size of angle TRU.

(a) From the diagram, $T(0,4,4)$ and $U(4,4,4)$.

(b) $$\overrightarrow{RT} = \underline{t} - \underline{r} = \begin{pmatrix} 0 \\ 4 \\ 4 \end{pmatrix} - \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -4 \\ 4 \\ 4 \end{pmatrix},$$

$$\overrightarrow{RU} = \underline{u} - \underline{r} = \begin{pmatrix} 4 \\ 4 \\ 4 \end{pmatrix} - \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 4 \end{pmatrix}.$$

(c) $$\begin{aligned} \cos T\hat{R}U &= \frac{\overrightarrow{RT}.\overrightarrow{RU}}{\left|\overrightarrow{RT}\right|\left|\overrightarrow{RU}\right|} \\ &= \frac{(-4\times 0)+(4\times 4)+(4\times 4)}{\sqrt{(-4)^2+4^2+4^2}\sqrt{0^2+4^2+4^2}} \\ &= \frac{32}{\sqrt{3\times 16}\sqrt{2\times 16}} \\ &= \frac{2}{\sqrt{6}} \end{aligned}$$

$$\begin{aligned} T\hat{R}U &= \cos^{-1}\left(\tfrac{2}{\sqrt{6}}\right) \\ &= 35.3° \text{ (to 1 d.p.)} \quad \text{(or 0.615 radians (to 3 d.p.))} \end{aligned}$$

13 Perpendicular Vectors

EF

If $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are perpendicular then $\underset{\sim}{a}.\underset{\sim}{b} = 0$.

This is because
$$\begin{aligned}\underset{\sim}{a}.\underset{\sim}{b} &= |\underset{\sim}{a}||\underset{\sim}{b}|\cos\theta \\ &= |\underset{\sim}{a}||\underset{\sim}{b}|\cos 90^\circ \qquad (\theta = 90^\circ \text{ since perpendicular}) \\ &= 0 \qquad (\text{since } \cos 90^\circ = 0)\end{aligned}$$

Conversely, if $\underset{\sim}{a}.\underset{\sim}{b} = 0$ then $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are perpendicular.

EXAMPLES

1. Two vectors are defined as $\underset{\sim}{a} = 4\underset{\sim}{i} + 2\underset{\sim}{j} - 5\underset{\sim}{k}$ and $\underset{\sim}{b} = 2\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k}$.

   Show that $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are perpendicular.

$$\begin{aligned}\underset{\sim}{a}.\underset{\sim}{b} &= a_1b_1 + a_2b_2 + a_3b_3 \\ &= (4\times 2) + (2\times 1) + ((-5)\times 2) \\ &= 8 + 2 - 10 \\ &= 0\end{aligned}$$

Since $\underset{\sim}{a}.\underset{\sim}{b} = 0$, $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are perpendicular.

2. $\overrightarrow{PQ} = \begin{pmatrix} 4 \\ a \\ 7 \end{pmatrix}$ and $\overrightarrow{RS} = \begin{pmatrix} 2 \\ -3 \\ a \end{pmatrix}$ where a is a constant.

   Given that $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ are perpendicular, find the value of a.

Since $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ are perpendicular,

$$\begin{aligned}\overrightarrow{PQ}.\overrightarrow{RS} &= 0 \\ 4\times 2 + (-3a) + 7a &= 0 \\ 8 - 3a + 7a &= 0 \\ 8 + 4a &= 0 \\ a &= -2.\end{aligned}$$

14 Properties of the Scalar Product EF

Some properties of the scalar product are as follows:

$$\underline{a}.\underline{b} = \underline{b}.\underline{a}$$

$$\underline{a}.(\underline{b}+\underline{c}) = \underline{a}.\underline{b} + \underline{a}.\underline{c} \qquad \text{(Expanding brackets)}$$

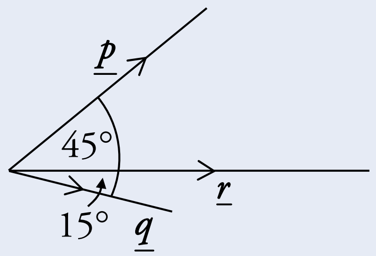
$$\underline{a}.\underline{a} = |\underline{a}|^2$$

Note that these are not given in the exam.

EXAMPLES

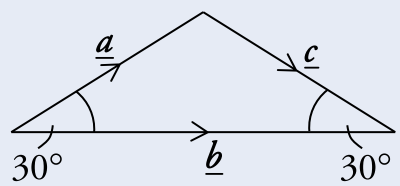
1. In the diagram, $|\underline{p}| = 3$, $|\underline{r}| = 4$ and $|\underline{q}| = 2$.

   Calculate $\underline{p}.(\underline{q}+\underline{r})$.



$$\begin{aligned}\underline{p}.(\underline{q}+\underline{r}) &= \underline{p}.\underline{q} + \underline{p}.\underline{r} \\ &= |\underline{p}||\underline{q}|\cos\theta_1 + |\underline{p}||\underline{r}|\cos\theta_2 \\ &= 3\times 2\times\cos 60° + 3\times 4\times\cos 45° \\ &= 6\times\tfrac{1}{2} + 12\times\tfrac{1}{\sqrt{2}} \\ &= 3 + 6\sqrt{2}.\end{aligned}$$

2. In the diagram below $|\underline{a}| = |\underline{c}| = 2$ and $|\underline{b}| = 2\sqrt{3}$.



   Calculate $\underline{a}.(\underline{a}+\underline{b}+\underline{c})$.

$$\begin{aligned}&\underline{a}.(\underline{a}+\underline{b}+\underline{c}) \\ &= \underline{a}.\underline{a} + \underline{a}.\underline{b} + \underline{a}.\underline{c} \\ &= |\underline{a}|^2 + |\underline{a}||\underline{b}|\cos\theta_1 - |\underline{a}||\underline{c}|\cos\theta_2 \\ &= 2^2 + 2\times 2\sqrt{3}\times\cos 30° - 2\times 2\times\cos 120° \\ &= 4 + 4\sqrt{3}\times\tfrac{\sqrt{3}}{2} + 4\times\tfrac{1}{2} \\ &= 4+6+2 \\ &= 12.\end{aligned}$$

Remember

$\underline{a}.\underline{c} = -|\underline{a}||\underline{c}|\cos\theta_2$ since $\underline{a}$ points to θ_2 and $\underline{c}$ points away.