Higher

# Mathematics

# Polynomials

## Contents



CfE Edition

## 8 Polynomials RC

Polynomials are expressions with one or more terms added together, where each term has a number (called the **coefficient**) followed by a variable (such as *x*) raised to a whole number power. For example:

$$3x^5 + x^3 + 2x^2 - 6 \quad \text{or} \quad 2x^{18} + 10.$$

The **degree** of the polynomial is the value of its highest power, for example:

$3x^5 + x^3 + 2x^2 - 6$ has degree 5 $\qquad$ $2x^{18} + 10$ has degree 18.

Note that quadratics are polynomials of degree two. Also, constants are polynomials of degree zero (e.g. 6 is a polynomial, since $6 = 6x^0$).

## 9 Synthetic Division

RC

Synthetic division provides a quick way of evaluating polynomials.

For example, consider $f(x) = 2x^3 - 9x^2 + 2x + 1$. Evaluating directly, we find $f(6) = 121$. We can also evaluate this using synthetic division with detached coefficients.

*Step 1*

Detach the coefficients, and write them across the top row of the table.

Note that they must be in order of *decreasing* degree. If there is no term of a specific degree, then zero is its coefficient.

| | | | | |
|---|---|---|---|---|
| | 2 | –9 | 2 | 1 |
| | | | | |
| | | | | |

*Step 2*

Write the number for which you want to evaluate the polynomial (the input number) to the left.

| | | | | |
|---|---|---|---|---|
| 6 | 2 | –9 | 2 | 1 |
| | | | | |
| | | | | |

*Step 3*

Bring down the first coefficient.

| | | | | |
|---|---|---|---|---|
| 6 | 2 | –9 | 2 | 1 |
| | ↓ | | | |
| | 2 | | | |

*Step 4*

Multiply this by the input number, writing the result underneath the next coefficient.

| | | | | |
|---|---|---|---|---|
| 6 | 2 | –9 | 2 | 1 |
| × | | 12 | | |
| | 2 | | | |

*Step 5*

Add the numbers in this column.

| | | | | |
|---|---|---|---|---|
| 6 | 2 | –9 | 2 | 1 |
| | | + 12 | | |
| | 2 | 3 | | |

Repeat Steps 4 and 5 until the last column has been completed.

The number in the lower-right cell is the value of the polynomial for the input value, often referred to as the **remainder**.

| | | | | | |
|---|---|---|---|---|---|
| 6 | 2 | –9 | 2 | 1 | |
| | | 12 | 18 | 120 | |
| | 2 | 3 | 20 | 121 | $= f(6)$ |

**EXAMPLE**

1. Given $f(x)=x^3+x^2-22x-40$, evaluate $f(-2)$ using synthetic division.

| | | | | |
|---|---|---|---|---|
| –2 | 1 | 1 | –22 | –40 |
| | | | | |
| | | | | |

using the above process →

| | | | | |
|---|---|---|---|---|
| –2 | 1 | 1 | –22 | –40 |
| | | –2 | 2 | 40 |
| | 1 | –1 | –20 | 0 |

So $f(-2)=0$.

## Note

In this example, the remainder is zero, so $f(-2)=0$.

This means $x^3+x^2-22x-40=0$ when $x=-2$, which means that $x=-2$ is a root of the equation. So $x+2$ must be a factor of the cubic.

We can use this to help with factorisation:

$f(x)=(x+2)(\ q(x)\ )$ where $q(x)$ is a quadratic

Is it possible to find the quadratic $q(x)$ using the table?

Trying the numbers from the bottom row as coefficients, we find:

$$(x+2)(x^2-x-20)$$
$$=x^3-x^2-20x+2x^2-2x-40$$
$$=x^3-x^2-22x-40$$
$$=f(x).$$

| | | | | |
|---|---|---|---|---|
| –2 | 1 | 1 | –22 | –40 |
| | | –2 | 2 | 40 |
| | 1 | –1 | –20 | 0 |

So using the numbers from the bottom row as coefficients has given the correct quadratic. In fact, this method *always* gives the correct quadratic, making synthetic division a useful tool for factorising polynomials.

**EXAMPLES**

2. Show that $x-4$ is a factor of $2x^4-9x^3+5x^2-3x-4$.

$x-4$ is a factor $\Leftrightarrow$ $x=4$ is a root.

| | | | | | |
|---|---|---|---|---|---|
| 4 | 2 | –9 | 5 | –3 | –4 |
| | | 8 | –4 | 4 | 4 |
| | 2 | –1 | 1 | 1 | 0 |

Since the remainder is zero, $x=4$ is a root, so $x-4$ is a factor.

3. Given $f(x) = x^3 - 37x + 84$, show that $x = -7$ is a root of $f(x) = 0$, and hence fully factorise $f(x)$.

$$\begin{array}{r|rrrr} -7 & 1 & 0 & -37 & 84 \\ & & -7 & 49 & -84 \\ \hline & 1 & -7 & 12 & \boxed{0} \end{array}$$

Since the remainder is zero, $x = -7$ is a root.

Hence we have
$$\begin{aligned} f(x) &= x^3 - 37x + 84 \\ &= (x+7)(x^2 - 7x + 12) \\ &= (x+7)(x-3)(x-4). \end{aligned}$$

4. Show that $x = -5$ is a root of $2x^3 + 7x^2 - 9x + 30 = 0$, and hence fully factorise the cubic.

$$\begin{array}{r|rrrr} -5 & 2 & 7 & -9 & 30 \\ & & -10 & 15 & -30 \\ \hline & 2 & -3 & 6 & \boxed{0} \end{array}$$

Since $x = -5$ is a root, $x + 5$ is a factor.

$$2x^3 + 7x^2 - 9x + 30 = (x+5)(2x^2 - 3x + 6)$$

This does not factorise any further since the quadratic has $b^2 - 4ac < 0$.

## Using synthetic division to factorise

In the examples above, we have been given a root or factor to help factorise polynomials. However, we can still use synthetic division if we do not know a factor or root.

Provided that the polynomial has an integer root, it will divide the constant term exactly. So by trying synthetic division with all divisors of the constant term, we will eventually find the integer root.

5. Fully factorise $2x^3 + 5x^2 - 28x - 15$.

Numbers which divide $-15$: $\pm 1, \pm 3, \pm 5, \pm 15$.

Try $x = 1$: 
$$\begin{aligned} & 2(1)^3 + 5(1)^2 - 28(1) - 15 \\ &= 2 + 5 - 28 - 15 \neq 0. \end{aligned}$$

Try $x = -1$: 
$$\begin{aligned} & 2(-1)^3 + 5(-1)^2 - 28(-1) - 15 \\ &= -2 + 5 + 28 - 15 \neq 0. \end{aligned}$$

**Note**
For ±1, it is simpler just to evaluate the polynomial directly, to see if these values are roots.

Try $x = 3$:

$$\begin{array}{r|rrrr} 3 & 2 & 5 & -28 & -15 \\ & & 6 & 33 & 15 \\ \hline & 2 & 11 & 5 & \boxed{0} \end{array}$$

Since $x = 3$ is a root, $x - 3$ is a factor.

$$\begin{aligned} \text{So } 2x^3 + 5x^2 - 28x - 15 &= (x-3)\left(2x^2 + 11x + 5\right) \\ &= (x-3)(2x+1)(x+5). \end{aligned}$$

## Using synthetic division to solve equations

We can also use synthetic division to help solve equations.

**EXAMPLE**

6. Find the solutions of $2x^3 - 15x^2 + 16x + 12 = 0$.

Numbers which divide 12: $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$.

Try $x = 1$: $2(1)^3 - 15(1)^2 + 16(1) + 12$

$= 2 - 15 + 16 + 12 \neq 0.$

Try $x = -1$: $2(-1)^3 - 15(-1)^2 + 16(-1) + 12$

$= -2 - 15 - 16 + 12 \neq 0.$

Try $x = 2$:

$$\begin{array}{r|rrrr} 2 & 2 & -15 & 16 & 12 \\ & & 4 & -22 & -12 \\ \hline & 2 & -11 & -6 & \boxed{0} \end{array}$$

Since $x = 2$ is a root, $x - 2$ is a factor:

$$\begin{aligned} 2x^3 - 15x^2 + 16x + 12 &= 0 \\ (x-2)\left(2x^2 - 11x - 6\right) &= 0 \\ (x-2)(2x+1)(x-6) &= 0 \end{aligned}$$

$x - 2 = 0$ or $2x + 1 = 0$ or $x - 6 = 0$

$x = 2$ $\quad x = -\frac{1}{2}$ $\quad x = 6.$

## The Factor Theorem and Remainder Theorem

For a polynomial $f(x)$:

If $f(x)$ is divided by $x - h$ then the remainder is $f(h)$, and

$$f(h) = 0 \Leftrightarrow x - h \text{ is a factor of } f(x).$$

As we saw, synthetic division helps us to write $f(x)$ in the form

$$(x-h)q(x) + f(h)$$

where $q(x)$ is called the **quotient** and $f(h)$ the **remainder**.

**EXAMPLE**

7. Find the quotient and remainder when $f(x)=4x^3+x^2-x-1$ is divided by $x+1$, and express $f(x)$ as $(x+1)q(x)+f(h)$.

| | | | | |
|---|---|---|---|---|
| $-1$ | 4 | 1 | $-1$ | $-1$ |
| | | $-4$ | 3 | $-2$ |
| | 4 | $-3$ | 2 | $-3$ |

The quotient is $4x^2-3x+2$ and the remainder is –3, so

$$f(x)=(x+1)\left(4x^2-3x+2\right)-3.$$

## 10 Finding Unknown Coefficients RC

Consider a polynomial with some unknown coefficients, such as $x^3+2px^2-px+4$, where $p$ is a constant.

If we divide the polynomial by $x-h$, then we will obtain an expression for the remainder in terms of the unknown constants. If we already know the value of the remainder, we can solve for the unknown constants.

**EXAMPLES**

1. Given that $x-3$ is a factor of $x^3-x^2+px+24$, find the value of $p$.

$x-3$ is a factor $\Leftrightarrow$ $x=3$ is a root.

| | | | | |
|---|---|---|---|---|
| 3 | 1 | $-1$ | $p$ | 24 |
| | | 3 | 6 | $18+3p$ |
| | 1 | 2 | $6+p$ | $42+3p$ |

**Note**
This is just the same synthetic division procedure we are used to.

Since $x=3$ is a root, the remainder is zero:

$$42+3p=0$$
$$3p=-42$$
$$p=-14.$$

2. When $f(x) = px^3 + qx^2 - 17x + 4q$ is divided by $x - 2$, the remainder is 6, and $x - 1$ is a factor of $f(x)$.

   Find the values of $p$ and $q$.

When $f(x)$ is divided by $x - 2$, the remainder is 6.

$$\begin{array}{c|cccc} 2 & p & q & -17 & 4q \\ & & 2p & 4p+2q & 8p+4q-34 \\ \hline & p & 2p+q & 4p+2q-17 & \boxed{8p+8q-34} \end{array}$$

Since the remainder is 6, we have:

$$\begin{aligned} 8p + 8q - 34 &= 6 \\ 8p + 8q &= 40 \\ p + q &= 5. \quad ① \end{aligned}$$

Since $x - 1$ is a factor, $f(1) = 0$:

$$\begin{aligned} f(1) &= p(1)^3 + q(1)^2 - 17(1) + 4q \\ &= p + q - 17 + 4q \\ &= p + 5q - 17 \end{aligned}$$

i.e. $p + 5q = 17$. ②

**Note**
There is no need to use synthetic division here, but you could if you wish.

Solving ① and ② simultaneously, we obtain:

②−①: $4q = 12$

$q = 3.$

Put $q = 3$ into ①: $p + 3 = 5$

$p = 2.$

Hence $p = 2$ and $q = 3$.

## 11 Finding Intersections of Curves RC

We have already met intersections of lines and parabolas in this outcome, but we were mainly interested in finding equations of tangents

We will now look at how to find the actual points of intersection – and not just for lines and parabolas; the technique works for any polynomials.

**EXAMPLES**

1. Find the points of intersection of the line $y = 4x - 4$ and the parabola $y = 2x^2 - 2x - 12$.

To find intersections, equate:

$$\begin{aligned} 2x^2 - 2x - 12 &= 4x - 4 \\ 2x^2 - 6x - 8 &= 0 \\ x^2 - 3x - 4 &= 0 \\ (x+1)(x-4) &= 0 \\ x = -1 \quad \text{or} \quad x &= 4. \end{aligned}$$

Find the *y*-coordinates by putting the *x*-values into one of the equations:

when $x = -1$, $y = 4 \times (-1) - 4 = -4 - 4 = -8$,
when $x = 4$, $y = 4 \times 4 - 4 = 16 - 4 = 12$.

So the points of intersection are $(-1,-8)$ and $(4,12)$.

2. Find the coordinates of the points of intersection of the cubic $y = x^3 - 9x^2 + 20x - 10$ and the line $y = -3x + 5$.

To find intersections, equate:

$$\begin{aligned} x^3 - 9x^2 + 20x - 10 &= -3x + 5 \\ x^3 - 9x^2 + 23x - 15 &= 0 \\ (x-1)(x^2 - 8x + 15) &= 0 \\ (x-1)(x-3)(x-5) &= 0 \\ x = 1 \quad \text{or} \quad x = 3 \quad \text{or} \quad x &= 5. \end{aligned}$$

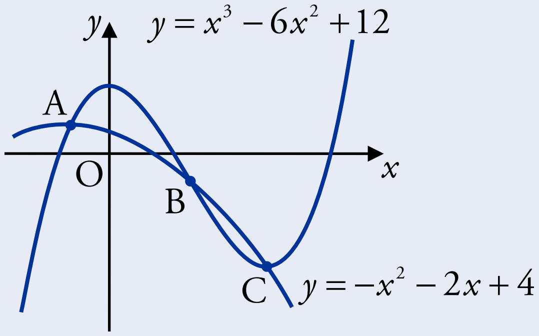
**Remember**
You can use synthetic division to help with factorising.

Find the *y*-coordinates by putting the *x*-values into one of the equations:

when $x = 1$, $y = -3 \times 1 + 5 = -3 + 5 = 2$,
when $x = 3$, $y = -3 \times 3 + 5 = -9 + 5 = -4$,
when $x = 5$, $y = -3 \times 5 + 5 = -15 + 5 = -10$.

So the points of intersection are $(1,2)$, $(3,-4)$ and $(5,-10)$.

3. The curves $y=-x^2-2x+4$ and $y=x^3-6x^2+12$ are shown below.




Find the $x$-coordinates of A, B and C, where the curves intersect.

To find intersections, equate:

$$-x^2-2x+4=x^3-6x^2+12$$
$$x^3-5x^2+2x+8=0$$
$$(x+1)\left(x^2-6x+8\right)=0$$
$$(x+1)(x-2)(x-4)=0$$
$$x=-1 \quad \text{or} \quad x=2 \quad \text{or} \quad x=4.$$

So at A, $x=-1$; at B, $x=2$; and at C, $x=4$.

**Remember**
You can use synthetic division to help with factorising.

4. Find the $x$-coordinates of the points where the curves $y=2x^3-3x^2-10$ and $y=3x^3-10x^2+7x+5$.

To find intersections, equate:

$$2x^3-3x^2-10=3x^3-10x^2+7x+5$$
$$x^3-7x^2+7x+15=0$$
$$(x+1)\left(x^2-8x+15\right)=0$$
$$(x+1)(x-3)(x-5)=0$$
$$x=-1 \quad \text{or} \quad x=3 \quad \text{or} \quad x=5.$$

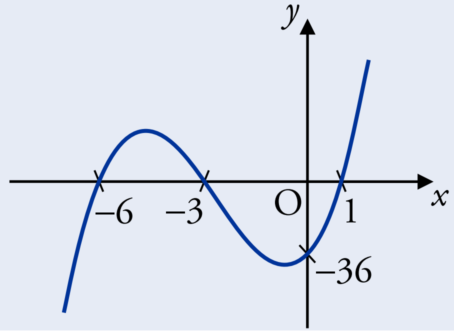
So the curves intersect where $x=-1,3,5$.

## 12 Determining the Equation of a Curve RC

Given the roots, and at least one other point lying on the curve, we can establish its equation using a process similar to that used when finding the equation of a parabola.

**EXAMPLE**

1. Find the equation of the cubic shown in the diagram below.




*Step 1*

Write out the roots, then rearrange to get the factors.

$$x=-6 \qquad x=-3 \qquad x=1$$
$$x+6=0 \qquad x+3=0 \qquad x-1=0.$$

*Step 2*

The equation then has these factors multiplied together with a constant, $k$.

$$y=k(x+6)(x+3)(x-1).$$

*Step 3*

Substitute the coordinates of a known point into this equation to find the value of $k$.

Using $(0, -36)$:

$$\begin{aligned} k(0+6)(0+3)(0-1) &= -36 \\ k(3)(-1)(6) &= -36 \\ -18k &= -36 \\ k &= 2. \end{aligned}$$

*Step 4*

Replace $k$ with this value in the equation.

$$\begin{aligned} y &= 2(x+6)(x+3)(x-1) \\ &= 2(x+3)\left(x^2+5x-6\right) \\ &= 2\left(x^3+5x^2-6x+3x^2+15x-18\right) \\ &= 2x^3+16x^2+18x-36. \end{aligned}$$

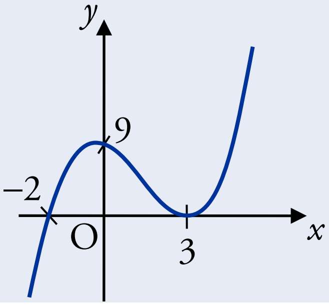
## Repeated Roots

If a repeated root exists, then a stationary point lies on the $x$-axis.

Recall that a repeated root exists when two roots, and hence two factors, are equal.

**EXAMPLE**

2. Find the equation of the cubic shown in the diagram below.




$$x = -2 \qquad x = 3 \qquad x = 3$$
$$x + 2 = 0 \qquad x - 3 = 0 \qquad x - 3 = 0.$$

So $y = k(x+2)(x-3)^2$.

Use $(0, 9)$ to find $k$:

$$9 = k(0+2)(0-3)^2$$
$$9 = k \times 2 \times 9$$
$$k = \tfrac{1}{2}.$$

So
$$\begin{aligned} y &= \tfrac{1}{2}(x+2)(x-3)^2 \\ &= \tfrac{1}{2}(x+2)\left(x^2 - 6x + 9\right) \\ &= \tfrac{1}{2}\left(x^3 - 6x^2 + 9x + 2x^2 - 12x + 18\right) \\ &= \tfrac{1}{2}x^3 - 2x^2 - \tfrac{3}{2}x + 9. \end{aligned}$$

**Note**

$x = 3$ is a repeated root, so the factor $(x - 3)$ appears twice in the equation.