Higher
Mathematics

Functions

Contents

CfE Edition

Functions and Graphs

1   Sets

In order to study functions and graphs, we use set theory. This requires some standard symbols and terms, which you should become familiar with.

A **set** is a collection of objects (usually numbers).

For example, $S = \{5,6,7,8\}$ is a set (we just list the object inside curly brackets).

We refer to the objects in a set as its **elements** (or members), e.g. 7 is an element of S. We can write this symbolically as $7 \in S$. It is also clear that 4 is *not* an element of S; we can write $4 \notin S$.

Given two sets A and B, we say A is a **subset** of B if all elements of A are also elements of B. For example, $\{6,7,8\}$ is a subset of S.

The **empty set** is the set with no elements. It is denoted by { } or $\varnothing$.

Standard Sets

There are common sets of numbers which have their own symbols. Note that numbers can belong to more than one set.

$\mathbb{N}$	natural numbers	counting numbers, i.e. $\mathbb{N} = \{1, 2, 3, 4, 5,\ldots\}$.
$\mathbb{Z}$	integers	positive and negative whole numbers, i.e. $\mathbb{Z} = \{\ldots,-2, -1, 0, 1, 2,\ldots\}$.
$\mathbb{Q}$	rational numbers	can be written as a fraction of integers, e.g. $-4, \frac{1}{3}, 0{\cdot}25, -\frac{1}{3}$.
$\mathbb{R}$	real numbers	all points on the number line, e.g. $-6, -\frac{1}{2}, \sqrt{2}, \frac{1}{12}, 0{\cdot}125$.

Notice that $\mathbb{N}$ is a subset of $\mathbb{Z}$, which is a subset of $\mathbb{Q}$, which is a subset of $\mathbb{R}$. These relationships between the standard sets are illustrated in the "Venn diagram" below.
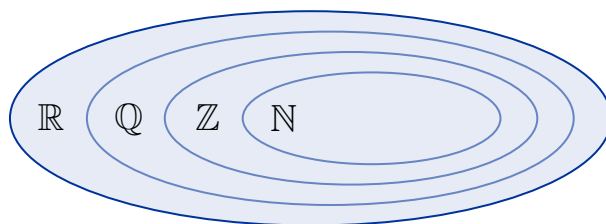
EXAMPLE

List all the numbers in the set $P = \{x \in \mathbb{N} : 1 < x < 5\}$.

P contains natural numbers which are strictly greater than 1 and strictly less than 5, so:
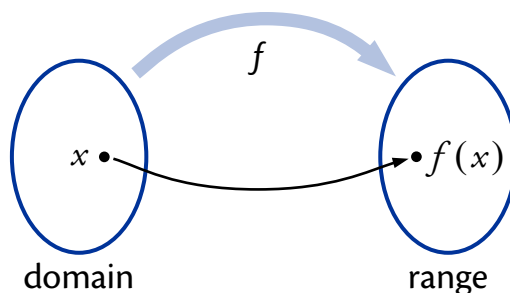
$$P = \{2, 3, 4\}.$$

Note

In set notation, a colon $(:)$ means "such that".

2   Functions   EF

A **function** relates a set of inputs to a set of outputs, with each input related to exactly one output.

The set of inputs is called the **domain** and the resulting set of outputs is called the **range**.

domain   range

A function is usually denoted by a lower case letter (e.g. f or g) and is defined using a formula of the form $f(x) = \ldots$. This specifies what the output of the function is when x is the input.

For example, if $f(x) = x^2 + 1$ then f squares the input and adds 1.

Restrictions on the Domain

The domain is the set of all possible inputs to a function, so it must be possible to evaluate the function for any element of the domain.

We are free to choose the domain, provided that the function is defined for all elements in it. If no domain is specified then we assume that it is as large as possible.

Division by Zero

It is impossible to divide by zero. So in functions involving fractions, the domain must exclude numbers which would give a denominator (bottom line) of zero.

For example, the function defined by:

$$f(x) = \frac{3}{x-5}$$

cannot have 5 in its domain, since this would make the denominator equal to zero.

The domain of f may be expressed formally as $\{x \in \mathbb{R} : x \neq 5\}$. This is read as "all x belonging to the real set such that x does not equal five".

Even Roots

Using real numbers, we cannot evaluate an even root (i.e. square root, fourth root etc.) of a negative number. So the domain of any function involving even roots must exclude numbers which would give a negative number under the root.

For example, the function defined by:

$$f(x) = \sqrt{7x-2}$$

must have $7x - 2 \geq 0$. Solving for x gives $x \geq \frac{2}{7}$, so the domain of f can be expressed formally as $\left\{x \in \mathbb{R} : x \geq \frac{2}{7}\right\}$.

EXAMPLE

1. A function g is defined by $g(x) = x - \dfrac{6}{x+4}$.

    Define a suitable domain for g.

    We cannot divide by zero, so $x \neq -4$. The domain is $\{x \in \mathbb{R} : x \neq -4\}$.
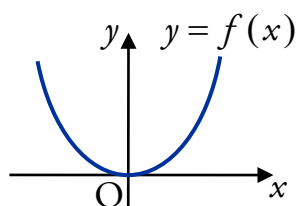
Identifying the Range

Recall that the range is the set of possible outputs. Some functions cannot produce certain values so these are not in the range.

For example:

$$f(x) = x^2$$

does not produce negative values, since any number squared is either positive or zero.

Looking at the graph of a function makes identifying its range more straightforward.

If we consider the graph of $y = f(x)$ (shown to the left) it is clear that there are no negative y-values.
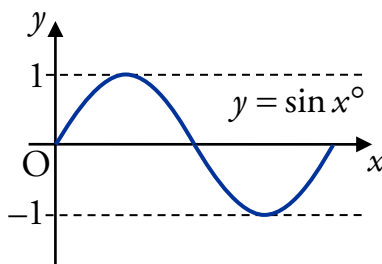
The range can be stated as $f(x) \geq 0$.

Note that the range also depends on the choice of domain. For example, if the domain of $f(x) = x^2$ is chosen to be $\{x \in \mathbb{R} : x \geq 3\}$ then the range can be stated as $f(x) \geq 9$.

EXAMPLE

2. A function f is defined by $f(x) = \sin x°$ for $x \in \mathbb{R}$. Identify its range.

Sketching the graph of $y = f(x)$ shows that $\sin x°$ only produces values from -1 to 1 inclusive.

This can be written as $-1 \leq f(x) \leq 1$.

3   Composite Functions

Two functions can be "composed" to form a new **composite function**.

For example, if we have a squaring function and a halving function, we can compose them to form a new function. We take the output from one and use it as the input for the other.

$$x \longrightarrow \boxed{\text{square}} \longrightarrow x^2 \longrightarrow \boxed{\text{halve}} \longrightarrow \frac{x^2}{2}$$
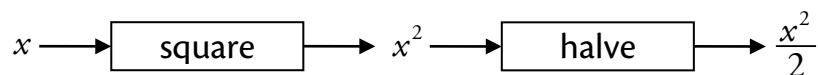
The order is important, as we get a different result in this case:

$$x \longrightarrow \boxed{\text{halve}} \longrightarrow \frac{x}{2} \longrightarrow \boxed{\text{square}} \longrightarrow \frac{x^2}{4}$$
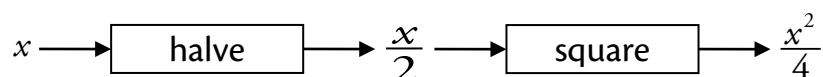
Using function notation we have, say, $f(x) = x^2$ and $g(x) = \frac{x}{2}$.

The diagrams above show the composite functions:

$$g(f(x)) = g(x^2) \qquad\qquad f(g(x)) = f\left(\frac{x}{2}\right)$$
$$= \frac{x^2}{2} \qquad\qquad\qquad = \left(\frac{x}{2}\right)^2 = \frac{x^2}{4}.$$

EXAMPLES

1. Functions f and g are defined by $f(x) = 2x$ and $g(x) = x - 3$. Find:
   (a) $f(2)$ (b) $f(g(x))$ (c) $g(f(x))$

   (a) $f(2) = 2(2)$ (b) $f(g(x)) = f(x-3)$ (c) $g(f(x)) = g(2x)$
   $\qquad = 4.$ $\qquad\qquad\qquad = 2(x-3).$ $\qquad\qquad\qquad = 2x - 3.$

2. Functions f and g are defined on suitable domains by $f(x) = x^3 + 1$ and $g(x) = \frac{1}{x}$.

   Find formulae for $h(x) = f(g(x))$ and $k(x) = g(f(x))$.

   $$h(x) = f(g(x)) \qquad\qquad k(x) = g(f(x))$$
   $$= f\left(\tfrac{1}{x}\right) \qquad\qquad\qquad = g(x^3 + 1)$$
   $$= \left(\tfrac{1}{x}\right)^3 + 1. \qquad\qquad = \frac{1}{x^3 + 1}.$$

4   Inverse Functions   EF

The idea of an **inverse function** is to reverse the effect of the original function. It is the "opposite" function.

You should already be familiar with this idea – for example, doubling a number can be reversed by halving the result. That is, multiplying by two and dividing by two are inverse functions.

The inverse of the function f is usually denoted f^{-1} (read as "f inverse").

> The functions f and g are said to be **inverses** if $f(g(x)) = g(f(x)) = x$.

This means that when a number is worked through a function f then its inverse f^{-1}, the result is the same as the input.
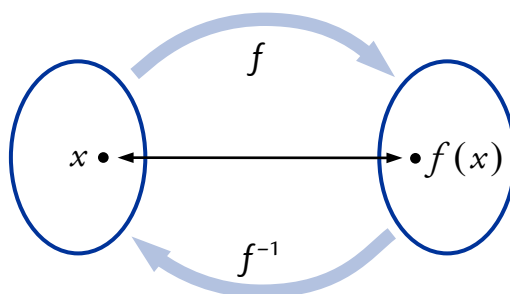
For example, $f(x) = 4x - 1$ and $g(x) = \dfrac{x+1}{4}$ are inverse functions since:

$$f(g(x)) = f\left(\frac{x+1}{4}\right) \qquad\qquad g(f(x)) = g(4x-1)$$

$$= 4\left(\frac{x+1}{4}\right) - 1 \qquad\qquad\quad = \frac{(4x-1)+1}{4}$$

$$= x + 1 - 1 \qquad\qquad\qquad\quad = \frac{4x}{4}$$

$$= x \qquad\qquad\qquad\qquad\qquad = x.$$

Formulae for Inverses

The example below shows how to find a formula for the inverse of a function.

EXAMPLES

1. A function f is defined, for all real numbers, by $f(x) = x^3 + 1$.

   Find a formula for its inverse f^{-1}.

Step 1

Replace $f(x)$ with y in the formula.   Let $y = x^3 + 1$.

Step 2

Change the subject of this
formula to x.

$$y = x^3 + 1$$
$$x^3 = y - 1$$
$$x = \sqrt[3]{y-1}.$$

Step 3

Interchange x and y.

$$y = \sqrt[3]{x-1}.$$

Step 4

Replace y with $f^{-1}(x)$ to obtain
the formula for the inverse.

$$f^{-1}(x) = \sqrt[3]{x-1}.$$

2. A function g is defined, for all real numbers, by $g(x) = \dfrac{x-3}{2}$.

   Find a formula for its inverse g^{-1}.
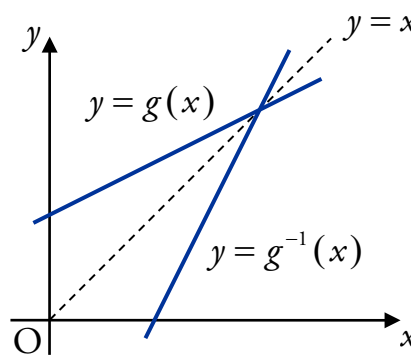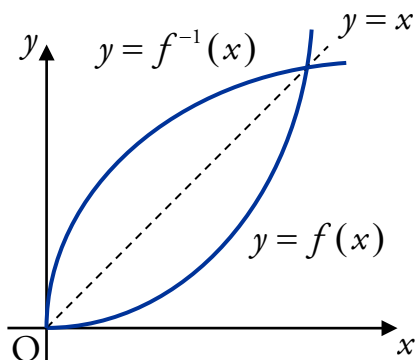
Let $y = \dfrac{x-3}{2}$.

Then $2y = x - 3$
$$x = 2y + 3.$$

So $g^{-1}(x) = 2x + 3$.

Graphs of Inverses

> If we have the graph of a function, then we can find the graph of its inverse by reflecting in the line $y = x$.

For example, the diagrams below show the graphs of two functions and their inverses.
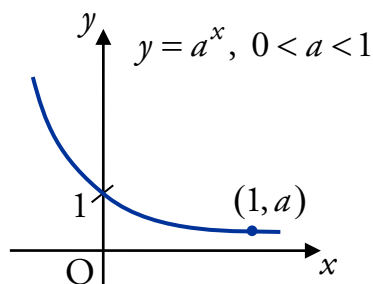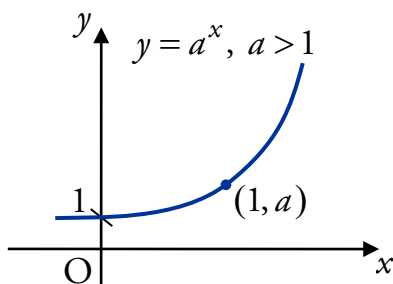
5 Exponential Functions   EF

> A function of the form $f(x) = a^x$ where $a, x \in \mathbb{R}$ and $a > 0$ is known as an **exponential** function to the base a.

We refer to x as the **power**, index or exponent.

Notice that when $x = 0$, $f(x) = a^0 = 1$. Also when $x = 1$, $f(x) = a^1 = a$.
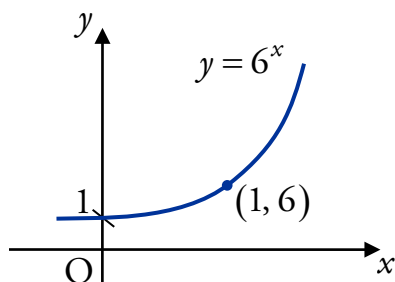
Hence the graph of an exponential always passes through $(0, 1)$ and $(1, a)$:

EXAMPLE

Sketch the curve with equation $y = 6^x$.

The curve passes through $(0, 1)$ and $(1, 6)$.

6   Introduction to Logarithms   EF

Until now, we have only been able to solve problems involving exponentials when we know the index, and have to find the base. For example, we can solve $p^6 = 512$ by taking sixth roots to get $p = \sqrt[6]{512}$.

But what if we know the base and have to find the index?

To solve $6^q = 512$ for q, we need to find the power of 6 which gives 512. To save writing this each time, we use the notation $q = \log_6 512$, read as "log to the base 6 of 512". In general:
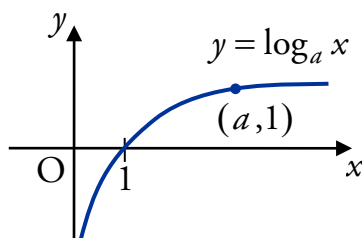
> $\log_a x$ is the power of a which gives x.

The properties of logarithms are covered in the notes on Exponentials and Logarithms.

Logarithmic Functions

> A **logarithmic** function is one in the form $f(x) = \log_a x$ where $a, x > 0$.

Logarithmic functions are inverses of exponentials, so to find the graph of $y = \log_a x$, we can reflect the graph of $y = a^x$ in the line $y = x$.

The graph of a logarithmic function always passes through $(1, 0)$ and $(a, 1)$.

EXAMPLE

Sketch the curve with equation $y = \log_6 x$.

The curve passes through $(1, 0)$ and $(6, 1)$.